(6.3) Curvilinear Coordinate Systems

The Navier Stokes equations are usually derived using cartesian coordinates; however, for many applications more general curvilinear coordinates systems are beneficial to both describe the flow geometry/boundaries and for ease in imposing the boundary conditions. For many analytical solutions orthogonal curvilinear coordinates are used whereas for CFD nonorthogonal coordinates are mostly used however some research CFD codes use orthogonal curvilinear coordinates. The transformation from cartesian to curvilinear coordinates can be done using both vector and tensor analysis. Here, a vector approach has been used with focus on orthogonal curvilinear coordinates as it lends itself to more physical insight. See Stern et al. (1986) and Richmond et al. (1986) for details of vector and tensor approaches for nonorthogonal curvilinear coordinates.

Outline:

1. Cartesian coordinates
2. Orthogonal curvilinear coordinate systems
3. Differential operators in orthogonal curvilinear coordinate systems
4. Derivatives of the unit vectors in orthogonal curvilinear coordinate systems
5. Incompressible N-S equations in orthogonal curvilinear coordinate systems
6. Example: Incompressible N-S equations in cylindrical polar systems
7. Overview extensions for nonorthogonal curvilinear coordinates

1. Cartesian Coordinates

The governing equations are usually derived using the most basic coordinate system, i.e., Cartesian coordinates:

$$\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{Laplacian} = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Example: incompressible flow equations with $\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$

$$
\begin{aligned}
&\nabla \cdot \mathbf{V} = 0 \\
&\rho \frac{D\mathbf{V}}{Dt} = -\nabla\left(p + \gamma z\right) + \mu \nabla^2 \mathbf{V} \\
&\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = -\nabla\left(p + \gamma z\right) + \mu \nabla^2 \mathbf{V} \\
&\rho\left[\frac{\partial \mathbf{V}}{\partial t} + \frac{1}{2}\nabla\left(\mathbf{V} \cdot \mathbf{V}\right) - \mathbf{V} \times \boldsymbol{\omega}\right] = -\nabla\left(p + \gamma z\right) + \mu\left[\nabla\left(\nabla \cdot \mathbf{V}\right) - \nabla \times \boldsymbol{\omega}\right] \\
&\boldsymbol{\omega} = \nabla \times \mathbf{V}
\end{aligned}
$$

$\nabla\left(\nabla \cdot \mathbf{V}\right) = 0$ is retained to keep the complete the vector identity for $\nabla^2\mathbf{V}$. Once the equations are expressed in vector invariant form (as above) they can be transformed into any convenient coordinate system through the use of appropriate definitions for the vector operators $\nabla, \nabla\cdot, \nabla\times$, and ∇^2[1]. Useful gradient vector differentiation formulas as follows[2].

$$
\begin{aligned}
&\nabla \cdot (f\mathbf{a}) = f(\nabla \cdot \mathbf{a}) + (\nabla f) \cdot \mathbf{a}, \\
&\nabla \times (f\mathbf{a}) = f(\nabla \times \mathbf{a}) + (\nabla f) \times \mathbf{a}, \\
&\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}), \\
&\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b}, \\
&\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}), \\
&\nabla \times (\nabla f) = 0, \\
&\nabla \cdot (\nabla \times \mathbf{a}) = 0, \\
&\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - (\nabla \cdot \nabla)\mathbf{a}, \\
&\nabla \cdot \mathbf{x} = 3, \\
&\nabla \times \mathbf{x} = 0. \\
&(\mathbf{a} \cdot \nabla)\mathbf{x} = \mathbf{a}.
\end{aligned}
$$

[1] Vectors and vector operators are independent of the reference frame, whereas their components are not.

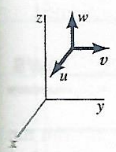
[2] Proofs are provided by G. E. Hay, Vector and Tensor Analysis, Dover Publications, Inc. 1953.

The Navier–Stokes Equations

The Navier–Stokes equations are the basic differential equations describing the flow of Newtonian fluids.

Incompressible
Constant property Flow
ie ρ + μ constant

For rectangular coordinates (see the figure in the margin) the results are



(x direction)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

(y direction)

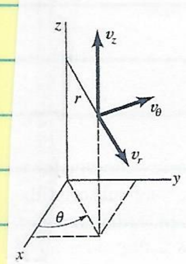
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

(z direction)

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

In terms of cylindrical polar coordinates (see the figure in the margin), the Navier–Stokes equations can be written as

$$\mathbf{V} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z$$



(r direction)

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

(θ direction)

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

(z direction)

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Continuity Equation

$$\nabla \cdot \mathbf{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

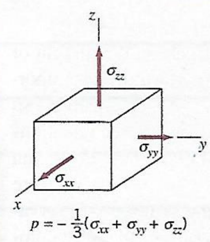
Stress–Deformation Relationships

For incompressible Newtonian fluids it is known that the stresses are linearly related to the rates of deformation and can be expressed in Cartesian coordinates as (for normal stresses)

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$



For Newtonian fluids, stresses are linearly related to the rate of strain.

(for shearing stresses)

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

In cylindrical polar coordinates the stresses for incompressible Newtonian fluids are expressed as (for normal stresses)

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

(for shearing stresses)

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

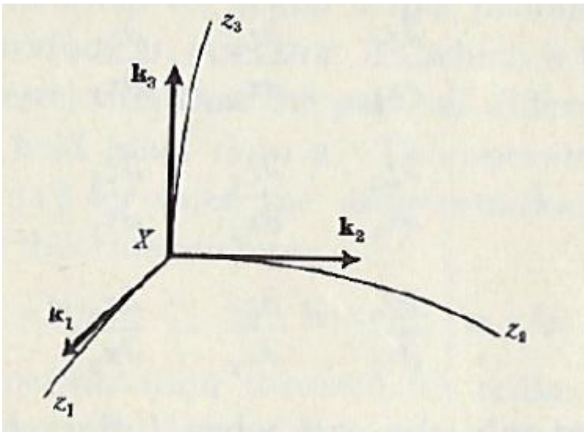
$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

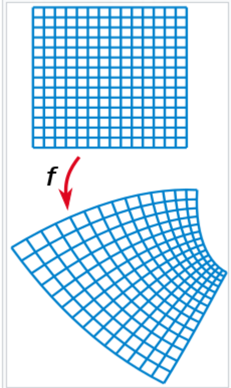
2. Orthogonal curvilinear coordinate systems

Suppose that the Cartesian coordinates (x, y, z) are expressed in terms of the new coordinates (x_1, x_2, x_3) by the equations

$$
\begin{aligned}
x &= x(x_1, x_2, x_3) \\
y &= y(x_1, x_2, x_3) \\
z &= z(x_1, x_2, x_3)
\end{aligned}
$$

where it is assumed that the correspondence is unique and that the inverse mapping exists.





A conformal map acting on a rectangular grid. Note that the orthogonality of the curved grid is retained.

Figures above show cartesian and orthogonal curvilinear coordinate systems and conformal mapping followed by table below of typical analytical orthogonal curvilinear coordinate systems from https://en.wikipedia.org/wiki/Orthogonal_coordinates.

For example, circular cylindrical coordinates $(x_1, x_2, x_3) = (r, \theta, z)$

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

i.e., at any point P, x_1 curve is a straight line, x_2 curve is a circle, and the x_3 curve is a straight line, i.e.

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1} y / x \\
z &= z
\end{aligned}
$$

Table of orthogonal coordinates [edit]

Besides the usual cartesian coordinates, several others are tabulated below.[5] Interval notation is used for compactness in the coordinates column.

Curvilinear coordinates (q_1, q_2, q_3)	Transformation from cartesian (*x*, *y*, *z*)	Scale factors
Spherical polar coordinates $(r,\theta,\phi)\in[0,\infty)\times[0,\pi]\times[0,2\pi)$	$x=r\sin\theta\cos\phi$ $y=r\sin\theta\sin\phi$ $z=r\cos\theta$	$h_1=1$ $h_2=r$ $h_3=r\sin\theta$
Cylindrical polar coordinates $(r,\phi,z)\in[0,\infty)\times[0,2\pi)\times(-\infty,\infty)$	$x=r\cos\phi$ $y=r\sin\phi$ $z=z$	$h_1=h_3=1$ $h_2=r$
Parabolic cylindrical coordinates $(u,v,z)\in(-\infty,\infty)\times[0,\infty)\times(-\infty,\infty)$	$x=\frac{1}{2}(u^2-v^2)$ $y=uv$ $z=z$	$h_1=h_2=\sqrt{u^2+v^2}$ $h_3=1$
Parabolic coordinates $(u,v,\phi)\in[0,\infty)\times[0,\infty)\times[0,2\pi)$	$x=uv\cos\phi$ $y=uv\sin\phi$ $z=\frac{1}{2}(u^2-v^2)$	$h_1=h_2=\sqrt{u^2+v^2}$ $h_3=uv$
Paraboloidal coordinates (λ,μ,ν) $\lambda<b^2<\mu<a^2<\nu$	$\frac{x^2}{q_i-a^2}+\frac{y^2}{q_i-b^2}=2z+q_i$ where $(q_1,q_2,q_3)=(\lambda,\mu,\nu)$	$h_i=\frac{1}{2}\sqrt{\frac{(q_j-q_i)(q_k-q_i)}{(a^2-q_i)(b^2-q_i)}}$
Ellipsoidal coordinates (λ,μ,ν) $\lambda<c^2<b^2<a^2,$ $c^2<\mu<b^2<a^2,$ $c^2<b^2<\nu<a^2,$	$\frac{x^2}{a^2-q_i}+\frac{y^2}{b^2-q_i}+\frac{z^2}{c^2-q_i}=1$ where $(q_1,q_2,q_3)=(\lambda,\mu,\nu)$	$h_i=\frac{1}{2}\sqrt{\frac{(q_j-q_i)(q_k-q_i)}{(a^2-q_i)(b^2-q_i)(c^2-q_i)}}$
Elliptic cylindrical coordinates $(u,v,z)\in[0,\infty)\times[0,2\pi)\times(-\infty,\infty)$	$x=a\cosh u\cos v$ $y=a\sinh u\sin v$ $z=z$	$h_1=h_2=a\sqrt{\sinh^2 u+\sin^2 v}$ $h_3=1$
Prolate spheroidal coordinates $(\xi,\eta,\phi)\in[0,\infty)\times[0,\pi]\times[0,2\pi)$	$x=a\sinh\xi\sin\eta\cos\phi$ $y=a\sinh\xi\sin\eta\sin\phi$ $z=a\cosh\xi\cos\eta$	$h_1=h_2=a\sqrt{\sinh^2\xi+\sin^2\eta}$ $h_3=a\sinh\xi\sin\eta$
Oblate spheroidal coordinates $(\xi,\eta,\phi)\in[0,\infty)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi)$	$x=a\cosh\xi\cos\eta\cos\phi$ $y=a\cosh\xi\cos\eta\sin\phi$ $z=a\sinh\xi\sin\eta$	$h_1=h_2=a\sqrt{\sinh^2\xi+\sin^2\eta}$ $h_3=a\cosh\xi\cos\eta$
Bipolar cylindrical coordinates $(u,v,z)\in[0,2\pi)\times(-\infty,\infty)\times(-\infty,\infty)$	$x=\frac{a\sinh v}{\cosh v-\cos u}$ $y=\frac{a\sin u}{\cosh v-\cos u}$ $z=z$	$h_1=h_2=\frac{a}{\cosh v-\cos u}$ $h_3=1$
Toroidal coordinates $(u,v,\phi)\in(-\pi,\pi]\times[0,\infty)\times[0,2\pi)$	$x=\frac{a\sinh v\cos\phi}{\cosh v-\cos u}$ $y=\frac{a\sinh v\sin\phi}{\cosh v-\cos u}$ $z=\frac{a\sin u}{\cosh v-\cos u}$	$h_1=h_2=\frac{a}{\cosh v-\cos u}$ $h_3=\frac{a\sinh v}{\cosh v-\cos u}$
Bispherical coordinates $(u,v,\phi)\in(-\pi,\pi]\times[0,\infty)\times[0,2\pi)$	$x=\frac{a\sin u\cos\phi}{\cosh v-\cos u}$ $y=\frac{a\sin u\sin\phi}{\cosh v-\cos u}$ $z=\frac{a\sinh v}{\cosh v-\cos u}$	$h_1=h_2=\frac{a}{\cosh v-\cos u}$ $h_3=\frac{a\sin u}{\cosh v-\cos u}$
Conical coordinates (λ,μ,ν) $\nu^2<b^2<\mu^2<a^2$ $\lambda\in[0,\infty)$	$x=\frac{\lambda\mu\nu}{ab}$ $y=\frac{\lambda}{a}\sqrt{\frac{(\mu^2-a^2)(\nu^2-a^2)}{a^2-b^2}}$ $z=\frac{\lambda}{b}\sqrt{\frac{(\mu^2-b^2)(\nu^2-b^2)}{b^2-a^2}}$	$h_1=1$ $h_2^2=\frac{\lambda^2(\mu^2-\nu^2)}{(\mu^2-a^2)(b^2-\mu^2)}$ $h_3^2=\frac{\lambda^2(\mu^2-\nu^2)}{(\nu^2-a^2)(\nu^2-b^2)}$

The position vector of a point P in space is

$$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\mathbf{R} = (r\cos\theta)\hat{\mathbf{i}} + (r\sin\theta)\hat{\mathbf{j}} + (z)\hat{\mathbf{k}} \text{ for cylindrical coordinates}$$

A vector tangent to the x_1 curve is given by:

$$\mathbf{R}_{x_1} = x_{x_1}\hat{\mathbf{i}} + y_{x_1}\hat{\mathbf{j}} + z_{x_1}\hat{\mathbf{k}} \text{ (Subscript denotes partial differentiation)}$$

$$\mathbf{R}_r = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$$

Similarly, for x_2 and x_3

$$\mathbf{R}_\theta = -r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}}$$

$$\mathbf{R}_z = \hat{\mathbf{k}}$$

So that the unit vectors tangent to the x_i curve are

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{R}_{x_1}}{h_1}, \qquad \hat{\mathbf{e}}_2 = \frac{\mathbf{R}_{x_2}}{h_2}, \qquad \hat{\mathbf{e}}_3 = \frac{\mathbf{R}_{x_3}}{h_3}$$

Where $h_i = \left|\mathbf{R}_{x_i}\right|$ are called the metric coefficients or scale factors:

$$h_r = 1, \qquad h_\theta = r, \qquad h_z = 1 \text{ for cylindrical coordinates}$$

The arc length along a curve in any direction is given by,

$$ds^2 = d\mathbf{R}\cdot d\mathbf{R} = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2$$

Since $d\mathbf{R} = \mathbf{R}_{x_i} dx_i = h_i dx_i \hat{\mathbf{e}}_i$, $\mathbf{R}_{x_i} = h_i \hat{\mathbf{e}}_i$ and x_i are orthogonal, i.e., $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

On the surface $x_1 =$ constant, the vector element of surface area is given by

$$d\mathbf{s}_1 = d\mathbf{R}_2 \times d\mathbf{R}_3 = h_2 dx_2 \hat{\mathbf{e}}_2 \times h_3 dx_3 \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 h_2 h_3 dx_2 dx_3$$

Where since x_i are orthogonal

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$$

and $-\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2$, $-\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3$ and $-\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1$ since $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

With similar results for x_2 and $x_3 =$ constant

$$d\mathbf{s}_2 = d\mathbf{R}_3 \times d\mathbf{R}_1 = \hat{\mathbf{e}}_2 h_3 h_1 dx_3 dx_1$$
$$d\mathbf{s}_3 = d\mathbf{R}_1 \times d\mathbf{R}_2 = \hat{\mathbf{e}}_3 h_1 h_2 dx_1 dx_2$$

An element of volume is given by the triple product

$$d\forall = d\mathbf{s}_3 \cdot d\mathbf{R}_3 = d\mathbf{R}_1 \times d\mathbf{R}_2 \cdot d\mathbf{R}_3 = \left(h_1 dx_1 \hat{\mathbf{e}}_1 \times h_2 dx_2 \hat{\mathbf{e}}_2\right) \cdot h_3 dx_3 \hat{\mathbf{e}}_3 = h_1 h_2 h_3 dx_1 dx_2 dx_3$$

$$d\forall = J\, dx_1 dx_2 dx_3$$

where J is called the Jacobian of the transformation of the variables. Also written

$$J = \frac{\partial(x, y, z)}{\partial(x_1, y_1, z_1)} = \begin{vmatrix} x_{x_1} & x_{x_2} & x_{x_3} \\ y_{x_1} & y_{x_2} & y_{x_3} \\ z_{x_1} & z_{x_2} & z_{x_3} \end{vmatrix}$$

Geometrically represents the ratio of volume elements dxdydz to $dx_1 dy_1 dz_1$ and a requirement for a conformal transformation

$$= x_{x_1}\left(y_{x_2} z_{x_3} - y_{x_3} z_{x_2}\right) - x_{x_2}\left(y_{x_1} z_{x_3} - y_{x_3} z_{x_1}\right) + x_{x_3}\left(y_{x_1} z_{x_2} - y_{x_2} z_{x_1}\right)$$

$= h_1 h_2 h_3$ for orthogonal curvilinear coordinates

3. Differential operators in orthogonal curvilinear coordinate systems

With the above in hand, we now proceed to obtain the desired vector operators.

3.1 Gradient $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\hat{\mathbf{e}}_3$

By definition: $df = \nabla f \cdot d\mathbf{R} = f_{x_i} dx_i$

If we temporarily write $\nabla f = \lambda_1\hat{\mathbf{e}}_1 + \lambda_2\hat{\mathbf{e}}_2 + \lambda_3\hat{\mathbf{e}}_3$ and using $d\mathbf{R} = \mathbf{R}_{x_i} dx_i = h_i dx_i \hat{\mathbf{e}}_i$

Then by comparison

$$df = f_{x_i} dx_i = \lambda_i h_i dx_i$$

$$\lambda_i = \frac{1}{h_i}\frac{\partial f}{\partial x_i}$$

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3$$

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial}{\partial z}\hat{\mathbf{e}}_z$$ for cylindrical coordinates

Note $\nabla x_i = \frac{\hat{\mathbf{e}}_i}{h_i} = \mathbf{R}_{x_i}$

By definition [$\mathrm{curl}(\mathrm{grad}\, f) = 0$]

$$\nabla \times \nabla x_i = \nabla \times \frac{\hat{\mathbf{e}}_i}{h_i} = 0$$

Also $\frac{\hat{\mathbf{e}}_1}{h_2 h_3} = \frac{\hat{\mathbf{e}}_2}{h_2} \times \frac{\hat{\mathbf{e}}_3}{h_3} = \nabla x_2 \times \nabla x_3$

By definition [$\nabla \cdot (\nabla f \times \nabla g) = \nabla g \cdot (\nabla \times \nabla f) - \nabla f \cdot (\nabla \times \nabla g) = 0$]

$$\nabla \cdot \left(\frac{\hat{\mathbf{e}}_1}{h_2 h_3}\right) = \nabla \cdot \left(\frac{\hat{\mathbf{e}}_2}{h_3 h_1}\right) = \nabla \cdot \left(\frac{\hat{\mathbf{e}}_3}{h_1 h_2}\right) = 0$$

3.2 Divergence $\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]$

$$\nabla \cdot \mathbf{F} = \nabla \cdot (F_1 \hat{\mathbf{e}}_1) + \nabla \cdot (F_2 \hat{\mathbf{e}}_2) + \nabla \cdot (F_3 \hat{\mathbf{e}}_3)$$

$$\nabla\cdot(F_1\hat{\mathbf{e}}_1)=\nabla\cdot\left[h_2h_3F_1\left(\frac{\hat{\mathbf{e}}_1}{h_2h_3}\right)\right]\text{ using }\nabla\cdot(\varphi\mathbf{u})=\varphi\nabla\cdot\mathbf{u}+\mathbf{u}\cdot\nabla\varphi$$

$$=\frac{\hat{\mathbf{e}}_1}{h_2h_3}\cdot\nabla(h_2h_3F_1)\text{ using }\nabla\cdot\left(\frac{\hat{\mathbf{e}}_1}{h_2h_3}\right)=\nabla\cdot\left(\frac{\hat{\mathbf{e}}_2}{h_3h_1}\right)=\nabla\cdot\left(\frac{\hat{\mathbf{e}}_3}{h_1h_2}\right)=0$$

$$=\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_1}(h_2h_3F_1)$$

Treating the other terms in a similar manner results in

$$\nabla\cdot\mathbf{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]$$

$$\nabla\cdot\mathbf{F}=\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1)+\frac{\partial}{\partial\theta}(F_2)+\frac{\partial}{\partial z}(rF_3)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial r}(rF_1)+\frac{1}{r}\frac{\partial}{\partial\theta}(F_2)+\frac{\partial}{\partial z}(F_3)\text{ for cylindrical coordinates}$$

3.3 Curl $\nabla\times\mathbf{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$

$$\nabla\times\mathbf{F}=\nabla\times(F_1\hat{\mathbf{e}}_1)+\nabla\times(F_2\hat{\mathbf{e}}_2)+\nabla\times(F_3\hat{\mathbf{e}}_3)$$

$$\nabla\times(F_1\hat{\mathbf{e}}_1)=\nabla\times\left[(h_1F_1)\left(\frac{\hat{\mathbf{e}}_1}{h_1}\right)\right]$$

$$=-\frac{\hat{\mathbf{e}}_1}{h_1}\times\nabla(h_1F_1)\text{ using }\nabla\times(\varphi\mathbf{u})=\varphi\nabla\times\mathbf{u}+\nabla\varphi\times\mathbf{u}\text{ along with }\nabla\times\frac{\hat{\mathbf{e}}_i}{h_i}=0\text{ and }\nabla\varphi\times\mathbf{u}=-\mathbf{u}\times\nabla\varphi$$

$$=-\frac{\hat{\mathbf{e}}_1}{h_1}\times\left[\frac{1}{h_1}\frac{\partial(h_1F_1)}{\partial x_1}\hat{\mathbf{e}}_1+\frac{1}{h_2}\frac{\partial(h_1F_1)}{\partial x_2}\hat{\mathbf{e}}_2+\frac{1}{h_3}\frac{\partial(h_1F_1)}{\partial x_3}\hat{\mathbf{e}}_3\right]$$

$$=-\frac{\hat{\mathbf{e}}_3}{h_1h_2}\frac{\partial}{\partial x_2}(h_1F_1)+\frac{\hat{\mathbf{e}}_2}{h_3h_1}\frac{\partial}{\partial x_3}(h_1F_1)$$

$$=\frac{1}{h_1h_2h_3}\left[h_2\hat{\mathbf{e}}_2\frac{\partial}{\partial x_3}-h_3\hat{\mathbf{e}}_3\frac{\partial}{\partial x_2}\right](h_1F_1)$$

$$\nabla\times\mathbf{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$$

$$\nabla\times\mathbf{F}=\frac{1}{r}\begin{vmatrix}\hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ F_1 & rF_2 & F_3\end{vmatrix}$$ for cylindrical coordinates

3.4 Laplacian acting on a scalar $\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial x_3}\right)\right]$

$$\nabla^2=\nabla\cdot\nabla=\nabla\cdot\frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1+\frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2+\frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\right)\right]$$

$$\nabla^2=\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)+\frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)$$ for cylindrical coordinates

3.5 Laplacian acting on a vector $\nabla^2\mathbf{F}=\nabla(\nabla\cdot\mathbf{F})-\nabla\times(\nabla\times\mathbf{F})$ **(vector identity)**

Using $\nabla=\frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1+\frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2+\frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3$

and $\nabla\cdot\mathbf{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]$

$\nabla(\nabla\cdot\mathbf{F})=$

$$\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_1$$

$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_2$$

$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_3$$

Using $\nabla\times\mathbf{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$

$$\nabla\times(\nabla\times\mathbf{F}) =$$

$$= \frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)\right]\hat{\mathbf{e}}_1$$
$$+\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)\right]\hat{\mathbf{e}}_2$$
$$+\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

Combining those two terms gives

$$\nabla^2\mathbf{F} = \nabla(\nabla\cdot\mathbf{F})-\nabla\times(\nabla\times\mathbf{F}) =$$

$$= \frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_1$$
$$-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)\right]\hat{\mathbf{e}}_1$$
$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_2$$
$$-\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)\right]\hat{\mathbf{e}}_2$$
$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_3$$
$$-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

For cylindrical coordinates (r,θ,z), $h_1 = h_r = 1,\ \ h_2 = h_\theta = r,\ \ h_3 = h_z = 1$, and use the definition of Laplacian operator acting on a scalar $\nabla^2 f$

$$\nabla^2 f = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right]$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$
$$= \frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$

$$\nabla^2\mathbf{F} = a\hat{\mathbf{e}}_r + b\hat{\mathbf{e}}_\theta + c\hat{\mathbf{e}}_z = \left(\nabla^2 F_1 - \frac{1}{r^2}F_1 - \frac{2}{r^2}\frac{\partial F_2}{\partial\theta}\right)\hat{\mathbf{e}}_r + \left(\nabla^2 F_2 - \frac{F_2}{r^2} + \frac{2}{r^2}\frac{\partial F_1}{\partial\theta}\right)\hat{\mathbf{e}}_\theta + \left(\nabla^2 F_3\right)\hat{\mathbf{e}}_z$$

$$a =$$

$$= \frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(h_2h_3F_1\right) + \frac{\partial}{\partial x_2}\left(h_3h_1F_2\right) + \frac{\partial}{\partial x_3}\left(h_1h_2F_3\right)\right]\right]$$

$$-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(h_2F_2\right) - \frac{\partial}{\partial x_2}\left(h_1F_1\right)\right]\right) - \frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(h_1F_1\right) - \frac{\partial}{\partial x_1}\left(h_3F_3\right)\right]\right)\right]$$

$$= \frac{\partial}{\partial r}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_1\right) + \frac{\partial}{\partial\theta}\left(F_2\right) + \frac{\partial}{\partial z}\left(rF_3\right)\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_2\right) - \frac{\partial}{\partial\theta}\left(F_1\right)\right]\right) - \frac{\partial}{\partial z}\left(r\left[\frac{\partial}{\partial z}\left(F_1\right) - \frac{\partial}{\partial r}\left(F_3\right)\right]\right)\right]$$

$$= \frac{\partial}{\partial r}\left[\frac{1}{r}\left[F_1 + r\frac{\partial F_1}{\partial r} + \frac{\partial F_2}{\partial\theta} + r\frac{\partial F_3}{\partial z}\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[F_2 + r\frac{\partial F_2}{\partial r} - \frac{\partial F_1}{\partial\theta}\right]\right) - \frac{\partial}{\partial z}\left(r\frac{\partial F_1}{\partial z} - r\frac{\partial F_3}{\partial r}\right)\right]$$

$$= \frac{\partial}{\partial r}\left[\frac{1}{r}F_1 + \frac{\partial F_1}{\partial r} + \frac{1}{r}\frac{\partial F_2}{\partial\theta} + \frac{\partial F_3}{\partial z}\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}F_2 + \frac{\partial F_2}{\partial r} - \frac{1}{r}\frac{\partial F_1}{\partial\theta}\right) - r\frac{\partial^2 F_1}{\partial z^2} + r\frac{\partial^2 F_3}{\partial z\partial r}\right]$$

$$= \left(\frac{-1}{r^2}F_1\right) + \frac{1}{r}\frac{\partial F_1}{\partial r} + \frac{\partial^2 F_1}{\partial r^2} + \left(\frac{-1}{r^2}\frac{\partial F_2}{\partial\theta}\right) + \frac{1}{r}\frac{\partial^2 F_2}{\partial r\partial\theta} + \frac{\partial^2 F_3}{\partial r\partial z}$$

$$-\frac{1}{r}\left[\frac{1}{r}\frac{\partial F_2}{\partial\theta} + \frac{\partial^2 F_2}{\partial\theta\partial r} - \frac{1}{r}\frac{\partial^2 F_1}{\partial\theta^2} - r\frac{\partial^2 F_1}{\partial z^2} + r\frac{\partial^2 F_3}{\partial z\partial r}\right]$$

$$= \frac{-1}{r^2}F_1 + \frac{1}{r}\frac{\partial F_1}{\partial r} + \frac{\partial^2 F_1}{\partial r^2} - \frac{1}{r^2}\frac{\partial F_2}{\partial\theta} + \cancel{\frac{1}{r}\frac{\partial^2 F_2}{\partial r\partial\theta}} + \cancel{\frac{\partial^2 F_3}{\partial r\partial z}}$$

$$+\left[-\frac{1}{r^2}\frac{\partial F_2}{\partial\theta} - \cancel{\frac{1}{r}\frac{\partial^2 F_2}{\partial\theta\partial r}} + \frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2} + \frac{\partial^2 F_1}{\partial z^2} - \cancel{\frac{\partial^2 F_3}{\partial z\partial r}}\right]$$

$$= \frac{-1}{r^2}F_1 + \frac{1}{r}\frac{\partial F_1}{\partial r} + \frac{\partial^2 F_1}{\partial r^2} - \frac{2}{r^2}\frac{\partial F_2}{\partial\theta} + \left[\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2} + \frac{\partial^2 F_1}{\partial z^2}\right]$$

$$= \left(\frac{1}{r}\frac{\partial F_1}{\partial r} + \frac{\partial^2 F_1}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2} + \frac{\partial^2 F_1}{\partial z^2}\right) - \frac{1}{r^2}F_1 - \frac{2}{r^2}\frac{\partial F_2}{\partial\theta}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial F_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2} + \frac{\partial^2 F_1}{\partial z^2}\right) - \frac{1}{r^2}F_1 - \frac{2}{r^2}\frac{\partial F_2}{\partial\theta}$$

$$=\nabla^2 F_1 - \frac{1}{r^2}F_1 - \frac{2}{r^2}\frac{\partial F_2}{\partial \theta}$$

$b =$

$$=\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(h_2 h_3 F_1\right)+\frac{\partial}{\partial x_2}\left(h_3 h_1 F_2\right)+\frac{\partial}{\partial x_3}\left(h_1 h_2 F_3\right)\right]\right]$$

$$-\frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}\left(h_3 F_3\right)-\frac{\partial}{\partial x_3}\left(h_2 F_2\right)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(h_2 F_2\right)-\frac{\partial}{\partial x_2}\left(h_1 F_1\right)\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_1\right)+\frac{\partial}{\partial \theta}\left(F_2\right)+\frac{\partial}{\partial z}\left(rF_3\right)\right]\right]$$

$$-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(F_3\right)-\frac{\partial}{\partial z}\left(rF_2\right)\right]\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_2\right)-\frac{\partial}{\partial \theta}\left(F_1\right)\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\left[F_1 + r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial \theta}+r\frac{\partial F_3}{\partial z}\right]\right]$$

$$-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial F_3}{\partial \theta}-\frac{\partial F_2}{\partial z}\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\left[F_2 + r\frac{\partial F_2}{\partial r}-\frac{\partial F_1}{\partial \theta}\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}F_1+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial \theta}+\frac{\partial F_3}{\partial z}\right]$$

$$-\left[\frac{1}{r}\frac{\partial^2 F_3}{\partial z \partial \theta}-\frac{\partial^2 F_2}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}F_2+\frac{\partial F_2}{\partial r}-\frac{1}{r}\frac{\partial F_1}{\partial \theta}\right)\right]$$

$$=\frac{1}{r}\left[\frac{1}{r}\frac{\partial F_1}{\partial \theta}+\frac{\partial^2 F_1}{\partial \theta \partial r}+\frac{1}{r}\frac{\partial^2 F_2}{\partial \theta^2}+\frac{\partial^2 F_3}{\partial \theta \partial z}\right]$$

$$-\left[\frac{1}{r}\frac{\partial^2 F_3}{\partial z \partial \theta}-\frac{\partial^2 F_2}{\partial z^2}-\left(\frac{-F_2}{r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}-\frac{-1}{r^2}\frac{\partial F_1}{\partial \theta}-\frac{1}{r}\frac{\partial^2 F_1}{\partial r \partial \theta}\right)\right]$$

$$=\frac{1}{r^2}\frac{\partial F_1}{\partial \theta}+\cancel{\frac{1}{r}\frac{\partial^2 F_1}{\partial \theta \partial r}}+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial \theta^2}+\cancel{\frac{1}{r}\frac{\partial^2 F_3}{\partial \theta \partial z}}$$

$$+\left[-\cancel{\frac{1}{r}\frac{\partial^2 F_3}{\partial z \partial \theta}}+\frac{\partial^2 F_2}{\partial z^2}-\frac{F_2}{r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r^2}\frac{\partial F_1}{\partial \theta}-\cancel{\frac{1}{r}\frac{\partial^2 F_1}{\partial r \partial \theta}}\right]$$

$$=\left(\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial \theta^2}+\frac{\partial^2 F_2}{\partial z^2}\right)-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial \theta}$$

$$=\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_2}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial \theta^2}+\frac{\partial^2 F_2}{\partial z^2}\right)-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial \theta}$$

$$=\nabla^2 F_2 - \frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial \theta}$$

$$
\begin{aligned}
c &= \\
&= \frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right] \\
&-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right] \\
&= \frac{\partial}{\partial z}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1)+\frac{\partial}{\partial \theta}(F_2)+\frac{\partial}{\partial z}(rF_3)\right]\right] \\
&-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\left[\frac{\partial}{\partial z}(F_1)-\frac{\partial}{\partial r}(F_3)\right]\right)-\frac{\partial}{\partial \theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial \theta}(F_3)-\frac{\partial}{\partial z}(rF_2)\right]\right)\right] \\
&= \frac{\partial}{\partial z}\left[\frac{1}{r}\left[F_1+r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial \theta}+r\frac{\partial F_3}{\partial z}\right]\right] \\
&-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial F_1}{\partial z}-r\frac{\partial F_3}{\partial r}\right)-\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial F_3}{\partial \theta}-\frac{\partial F_2}{\partial z}\right)\right] \\
&= \frac{\partial}{\partial z}\left[\frac{F_1}{r}+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial \theta}+\frac{\partial F_3}{\partial z}\right] \\
&-\frac{1}{r}\left[\left(\frac{\partial F_1}{\partial z}+r\frac{\partial^2 F_1}{\partial r\partial z}-\frac{\partial F_3}{\partial r}-r\frac{\partial^2 F_3}{\partial r^2}\right)-\left(\frac{1}{r}\frac{\partial^2 F_3}{\partial \theta^2}-\frac{\partial^2 F_2}{\partial \theta\partial z}\right)\right] \\
&= \cancel{\frac{1}{r}\frac{\partial F_1}{\partial z}}+\cancel{\frac{\partial^2 F_1}{\partial z\partial r}}+\cancel{\frac{1}{r}\frac{\partial^2 F_2}{\partial z\partial \theta}}+\frac{\partial^2 F_3}{\partial z^2} \\
&+\left[-\cancel{\frac{1}{r}\frac{\partial F_1}{\partial z}}-\cancel{\frac{\partial^2 F_1}{\partial r\partial z}}+\frac{1}{r}\frac{\partial F_3}{\partial r}+\frac{\partial^2 F_3}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_3}{\partial \theta^2}-\cancel{\frac{1}{r}\frac{\partial^2 F_2}{\partial \theta\partial z}}\right] \\
&= \frac{1}{r}\frac{\partial F_3}{\partial r}+\frac{\partial^2 F_3}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_3}{\partial \theta^2}+\frac{\partial^2 F_3}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_3}{\partial r}\right)+\frac{\partial^2 F_3}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_3}{\partial \theta^2}+\frac{\partial^2 F_3}{\partial z^2} \\
&= \nabla^2 F_3
\end{aligned}
$$

4. Derivatives of the unit vectors in orthogonal curvilinear coordinate systems

The last topic to be discussed concerning curvilinear coordinates is the procedure to obtain the derivatives of the unit vectors, i.e. $\frac{\partial}{\partial \boldsymbol{x}_j}\hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_{i,j} = \hat{\boldsymbol{e}}_{ij}$ where the comma is dropped in the rest of the document.

Such quantities are required, for example, in obtaining the rate-of-strain and rotation tensor.

$$e_{ij} = \nabla \mathbf{V}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(e_{ij}+e_{ij}^T\right) = \frac{1}{2}\left(\nabla\mathbf{V}+\mathbf{V}\nabla\right)$$

$$\omega_{ij} = \frac{1}{2}\left(e_{ij} - e_{ij}^T\right) = \frac{1}{2}\left(\nabla\mathbf{V} - \mathbf{V}\nabla\right)$$

To simplify the notation, we define:

$$\mathbf{R}_{x_i} = \mathbf{r}_i, \quad \mathbf{R}_{x_i} = h_i\hat{\mathbf{e}}_i = \mathbf{r}_i$$

$$\frac{\partial}{\partial x_j}\mathbf{R}_{x_i} = \mathbf{r}_{ij} \text{ and } \frac{\partial}{\partial x_j}h_i = h_{ij}$$

Note that $\mathbf{r}_{ij}$ is symmetric, i.e. $\mathbf{r}_{ij} = \mathbf{r}_{ji}$

$$\begin{aligned}
\mathbf{r}_1 &= h_1\hat{\mathbf{e}}_1 \\
\mathbf{r}_2 &= h_2\hat{\mathbf{e}}_2 \\
\mathbf{r}_3 &= h_3\hat{\mathbf{e}}_3 \\
\mathbf{r}_{11} &= a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2 + c\hat{\mathbf{e}}_3 = h_{11}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{11} \\
\mathbf{r}_{12} &= h_{12}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{12} \\
\mathbf{r}_{13} &= h_{13}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{13}
\end{aligned}$$

4.1 Derivation of $\hat{\mathbf{e}}_{11} = -\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_{13}}{h_3}\hat{\mathbf{e}}_3$

$$\begin{aligned}
&\mathbf{r}_1 \cdot \mathbf{r}_1 = h_1^2 \\
&\mathbf{r}_1 \cdot \mathbf{r}_{11} = h_1 h_{11} = a h_1 \to a = h_{11} \\
&\mathbf{r}_1 \cdot \mathbf{r}_{12} = h_1 h_{12} \\
&\mathbf{r}_1 \cdot \mathbf{r}_{13} = h_1 h_{13}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{r}_1 \cdot \mathbf{r}_2 = 0 \\
&\to \frac{\partial\left(\mathbf{r}_1 \cdot \mathbf{r}_2\right)}{\partial x_1} = 0 \\
&\to \mathbf{r}_{11} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{21} = 0 \\
&\to \mathbf{r}_{11} \cdot \mathbf{r}_2 = -\mathbf{r}_1 \cdot \mathbf{r}_{12} \\
&\to \mathbf{r}_{11} \cdot \mathbf{r}_2 = -h_1 h_{12} = b h_2 \to b = -\frac{h_1 h_{12}}{h_2}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{r}_1 \cdot \mathbf{r}_3 = 0 \\
&\to \frac{\partial\left(\mathbf{r}_1 \cdot \mathbf{r}_3\right)}{\partial x_1} = 0 \\
&\to \mathbf{r}_{11} \cdot \mathbf{r}_3 + \mathbf{r}_1 \cdot \mathbf{r}_{31} = 0 \\
&\to \mathbf{r}_{11} \cdot \mathbf{r}_3 = -\mathbf{r}_1 \cdot \mathbf{r}_{13} \\
&\to \mathbf{r}_{11} \cdot \mathbf{r}_3 = -h_1 h_{13} = c h_3 \to c = -\frac{h_1 h_{13}}{h_3}
\end{aligned}$$

$$\mathbf{r}_{11} = h_{11}\hat{\mathbf{e}}_1 - \frac{h_1 h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_1 h_{13}}{h_3}\hat{\mathbf{e}}_3 = h_{11}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{11}$$

$$\rightarrow \hat{\mathbf{e}}_{11} = -\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_{13}}{h_3}\hat{\mathbf{e}}_3$$

4.2 Derivation of $\hat{\mathbf{e}}_{22} = -\frac{h_{21}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{23}}{h_3}\hat{\mathbf{e}}_3$

$$\mathbf{r}_2 \cdot \mathbf{r}_2 = h_2^2$$
$$\mathbf{r}_2 \cdot \mathbf{r}_{22} = h_2 h_{22}$$
$$\mathbf{r}_2 \cdot \mathbf{r}_{21} = h_2 h_{21}$$
$$\mathbf{r}_2 \cdot \mathbf{r}_{23} = h_2 h_{23}$$

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_1 \cdot \mathbf{r}_2)}{\partial x_2} = 0$$
$$\rightarrow \mathbf{r}_{12} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{22} = 0$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_1 = -\mathbf{r}_2 \cdot \mathbf{r}_{21}$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_1 = -h_2 h_{21}$$

$$\mathbf{r}_2 \cdot \mathbf{r}_3 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_2 \cdot \mathbf{r}_3)}{\partial x_2} = 0$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_3 + \mathbf{r}_2 \cdot \mathbf{r}_{32} = 0$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_3 = -\mathbf{r}_2 \cdot \mathbf{r}_{23}$$
$$\rightarrow \mathbf{r}_{22} \cdot \mathbf{r}_3 = -h_2 h_{23}$$

$$\mathbf{r}_{22} = -\frac{h_2 h_{21}}{h_1}\hat{\mathbf{e}}_1 + h_{22}\hat{\mathbf{e}}_2 - \frac{h_2 h_{23}}{h_3}\hat{\mathbf{e}}_3 = h_{22}\hat{\mathbf{e}}_2 + h_2\hat{\mathbf{e}}_{22}$$

$$\rightarrow \hat{\mathbf{e}}_{22} = -\frac{h_{21}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{23}}{h_3}\hat{\mathbf{e}}_3$$

4.3 Derivation of $\hat{\mathbf{e}}_{33} = -\frac{h_{31}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{32}}{h_2}\hat{\mathbf{e}}_2$

$$\mathbf{r}_3 \cdot \mathbf{r}_3 = h_3^2$$
$$\mathbf{r}_3 \cdot \mathbf{r}_{31} = h_3 h_{31}$$
$$\mathbf{r}_3 \cdot \mathbf{r}_{32} = h_3 h_{32}$$

$$\mathbf{r}_3 \cdot \mathbf{r}_{33} = h_3 h_{33}$$

$$\mathbf{r}_1 \cdot \mathbf{r}_3 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_1 \cdot \mathbf{r}_3)}{\partial x_3} = 0$$
$$\rightarrow \mathbf{r}_{13} \cdot \mathbf{r}_3 + \mathbf{r}_1 \cdot \mathbf{r}_{33} = 0$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_1 = -\mathbf{r}_3 \cdot \mathbf{r}_{31}$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_1 = -h_3 h_{31}$$

$$\mathbf{r}_2 \cdot \mathbf{r}_3 = 0$$
$$\rightarrow \frac{\partial(\mathbf{r}_2 \cdot \mathbf{r}_3)}{\partial x_3} = 0$$
$$\rightarrow \mathbf{r}_{23} \cdot \mathbf{r}_3 + \mathbf{r}_2 \cdot \mathbf{r}_{33} = 0$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_2 = -\mathbf{r}_3 \cdot \mathbf{r}_{32}$$
$$\rightarrow \mathbf{r}_{33} \cdot \mathbf{r}_2 = -h_3 h_{32}$$

$$\mathbf{r}_{33} = -\frac{h_3 h_{31}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_3 h_{32}}{h_2}\hat{\mathbf{e}}_2 + h_{33}\hat{\mathbf{e}}_3 = h_{33}\hat{\mathbf{e}}_3 + h_3\hat{\mathbf{e}}_{33}$$
$$\rightarrow \hat{\mathbf{e}}_{33} = -\frac{h_{31}}{h_1}\hat{\mathbf{e}}_1 - \frac{h_{32}}{h_2}\hat{\mathbf{e}}_2$$

4.4 Derivation of $\hat{\mathbf{e}}_{32} = \frac{h_{23}}{h_3}\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_{23} = \frac{h_{32}}{h_2}\hat{\mathbf{e}}_3$

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0 \rightarrow \frac{\partial}{\partial x_3}(\mathbf{r}_1 \cdot \mathbf{r}_2) = \mathbf{r}_{13} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{23} = 0 \qquad (1)$$
$$\mathbf{r}_2 \cdot \mathbf{r}_3 = 0 \rightarrow \frac{\partial}{\partial x_1}(\mathbf{r}_2 \cdot \mathbf{r}_3) = \mathbf{r}_{21} \cdot \mathbf{r}_3 + \mathbf{r}_2 \cdot \mathbf{r}_{31} = 0 \qquad (2)$$
$$\mathbf{r}_3 \cdot \mathbf{r}_1 = 0 \rightarrow \frac{\partial}{\partial x_2}(\mathbf{r}_3 \cdot \mathbf{r}_1) = \mathbf{r}_{32} \cdot \mathbf{r}_1 + \mathbf{r}_3 \cdot \mathbf{r}_{12} = 0 \qquad (3)$$

$$(3) - (2) = 0$$
$$\rightarrow \mathbf{r}_{32} \cdot \mathbf{r}_1 - \mathbf{r}_2 \cdot \mathbf{r}_{31} = 0$$
$$\rightarrow \mathbf{r}_{32} \cdot \mathbf{r}_1 = \mathbf{r}_2 \cdot \mathbf{r}_{31}$$
$$\rightarrow \mathbf{r}_{32} \cdot \mathbf{r}_1 = -\mathbf{r}_1 \cdot \mathbf{r}_{23}$$
$$\rightarrow \mathbf{r}_{32} \cdot \mathbf{r}_1 = 0$$

$$\mathbf{r}_{32} = a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2 + c\hat{\mathbf{e}}_3 = h_{32}\hat{\mathbf{e}}_3 + h_3\hat{\mathbf{e}}_{32}$$

$$\mathbf{r}_{32} = a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2 + c\hat{\mathbf{e}}_3 = h_{23}\hat{\mathbf{e}}_2 + h_{32}\hat{\mathbf{e}}_3$$

$$\hat{\mathbf{e}}_{32} = \frac{h_{23}}{h_3}\hat{\mathbf{e}}_2$$

$$\mathbf{r}_{23} = h_{23}\hat{\mathbf{e}}_2 + h_2\hat{\mathbf{e}}_{23}$$

$$\mathbf{r}_{23} = \mathbf{r}_{32} = a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2 + c\hat{\mathbf{e}}_3 = h_{23}\hat{\mathbf{e}}_2 + h_{32}\hat{\mathbf{e}}_3$$

$$\hat{\mathbf{e}}_{23} = \frac{h_{32}}{h_2}\hat{\mathbf{e}}_3$$

4.5 Derivation of $\hat{\mathbf{e}}_{21} = \frac{h_{12}}{h_2}\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_{12} = \frac{h_{21}}{h_1}\hat{\mathbf{e}}_2$

$$(2)-(1)=0$$

$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3 - \mathbf{r}_1\cdot\mathbf{r}_{23} = 0$$

$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3 = \mathbf{r}_1\cdot\mathbf{r}_{23}$$

$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3 = \mathbf{r}_3\cdot\mathbf{r}_{12}$$

$$\rightarrow \mathbf{r}_{21}\cdot\mathbf{r}_3 = 0$$

$$\mathbf{r}_{21} = h_{21}\hat{\mathbf{e}}_2 + h_2\hat{\mathbf{e}}_{21}$$

$$\mathbf{r}_{21} = h_{12}\hat{\mathbf{e}}_1 + h_{21}\hat{\mathbf{e}}_2$$

$$\hat{\mathbf{e}}_{21} = \frac{h_{12}}{h_2}\hat{\mathbf{e}}_1$$

$$\mathbf{r}_{12} = h_{12}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{12}$$

$$\mathbf{r}_{12} = \mathbf{r}_{21} = h_{12}\hat{\mathbf{e}}_1 + h_{21}\hat{\mathbf{e}}_2$$

$$\hat{\mathbf{e}}_{12} = \frac{h_{21}}{h_1}\hat{\mathbf{e}}_2$$

4.6 Derivation of $\hat{\mathbf{e}}_{13} = \frac{h_{31}}{h_1}\hat{\mathbf{e}}_3$, $\hat{\mathbf{e}}_{31} = \frac{h_{13}}{h_3}\hat{\mathbf{e}}_1$

$$(3)-(1)=0$$

$$\rightarrow \mathbf{r}_3\cdot\mathbf{r}_{12} - \mathbf{r}_{13}\cdot\mathbf{r}_2 = 0$$

$$\rightarrow \mathbf{r}_2\cdot\mathbf{r}_{13} = \mathbf{r}_3\cdot\mathbf{r}_{12}$$

$$\rightarrow \mathbf{r}_2\cdot\mathbf{r}_{13} = -\mathbf{r}_2\cdot\mathbf{r}_{31}$$

$$\rightarrow \mathbf{r}_2\cdot\mathbf{r}_{13} = 0$$

$$\mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{13}$$

$$\mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_{31}\hat{\mathbf{e}}_3$$

$$\hat{\mathbf{e}}_{13} = \frac{h_{31}}{h_1}\hat{\mathbf{e}}_3$$

$$\mathbf{r}_{31} = h_{31}\hat{\mathbf{e}}_3 + h_3\hat{\mathbf{e}}_{31}$$
$$\mathbf{r}_{31} = \mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_{31}\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_{31} = \frac{h_{13}}{h_3}\hat{\mathbf{e}}_1$$

5. Incompressible N-S equations in orthogonal curvilinear coordinate systems

5.1 Continuity equation $\nabla \cdot \mathbf{V} = 0$

Since $\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]$

and $\mathbf{V} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 v_1) + \frac{\partial}{\partial x_2}(h_3 h_1 v_2) + \frac{\partial}{\partial x_3}(h_1 h_2 v_3)\right] = 0$$

5.2 Momentum equation $\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V}$, (where p piezometric pressure)

Since $\mathbf{V} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$, we can expand the momentum equation term by term

Local derivative $\frac{\partial \mathbf{V}}{\partial t} = \frac{\partial v_1}{\partial t}\hat{\mathbf{e}}_1 + \frac{\partial v_2}{\partial t}\hat{\mathbf{e}}_2 + \frac{\partial v_3}{\partial t}\hat{\mathbf{e}}_3$

Convective derivative $(\mathbf{V}\cdot\nabla)\mathbf{V}$

Since $\mathbf{V} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$ and $\mathbf{V}\cdot\nabla = \frac{v_1}{h_1}\frac{\partial}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial}{\partial x_3}$

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = (\mathbf{V}\cdot\nabla)(v_1\hat{\mathbf{e}}_1) + (\mathbf{V}\cdot\nabla)(v_2\hat{\mathbf{e}}_2) + (\mathbf{V}\cdot\nabla)(v_3\hat{\mathbf{e}}_3)$$

$$(\mathbf{V}\cdot\nabla)(v_1\hat{\mathbf{e}}_1) =$$

$$= \frac{v_1}{h_1}\frac{\partial(v_1\hat{\mathbf{e}}_1)}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial(v_1\hat{\mathbf{e}}_1)}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial(v_1\hat{\mathbf{e}}_1)}{\partial x_3}$$

$$= \frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{v_1 v_1}{h_1}\frac{\partial\hat{\mathbf{e}}_1}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}\hat{\mathbf{e}}_1 + \frac{v_2 v_1}{h_2}\frac{\partial\hat{\mathbf{e}}_1}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\hat{\mathbf{e}}_1 + \frac{v_3 v_1}{h_3}\frac{\partial\hat{\mathbf{e}}_1}{\partial x_3}$$

$$= \left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \left(\frac{v_1 v_1}{h_1}\hat{\mathbf{e}}_{11} + \frac{v_2 v_1}{h_2}\hat{\mathbf{e}}_{12} + \frac{v_3 v_1}{h_3}\hat{\mathbf{e}}_{13}\right)$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1+\frac{v_1 v_1}{h_1}\left(-\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2-\frac{h_{13}}{h_3}\hat{\mathbf{e}}_3\right)$$

$$+\frac{v_2 v_1}{h_2}\left(\frac{h_{21}}{h_1}\hat{\mathbf{e}}_2\right)+\frac{v_3 v_1}{h_3}\left(\frac{h_{31}}{h_1}\hat{\mathbf{e}}_3\right)$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1+\left(\frac{v_2 v_1 h_{21}}{h_1 h_2}-\frac{v_1 v_1 h_{12}}{h_1 h_2}\right)\hat{\mathbf{e}}_2+\left(\frac{v_3 v_1 h_{31}}{h_3 h_1}-\frac{v_1 v_1 h_{13}}{h_3 h_1}\right)\hat{\mathbf{e}}_3$$

$$(\mathbf{V}\cdot\nabla)(v_2\hat{\mathbf{e}}_2)=$$

$$=\frac{v_1}{h_1}\frac{\partial(v_2\hat{\mathbf{e}}_2)}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial(v_2\hat{\mathbf{e}}_2)}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial(v_2\hat{\mathbf{e}}_2)}{\partial x_3}$$

$$=\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}\hat{\mathbf{e}}_2+\frac{v_1 v_2}{h_1}\frac{\partial\hat{\mathbf{e}}_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}\hat{\mathbf{e}}_2+\frac{v_2 v_2}{h_2}\frac{\partial\hat{\mathbf{e}}_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\hat{\mathbf{e}}_2+\frac{v_3 v_2}{h_3}\frac{\partial\hat{\mathbf{e}}_2}{\partial x_3}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2+\frac{v_1 v_2}{h_1}\hat{\mathbf{e}}_{21}+\frac{v_2 v_2}{h_2}\hat{\mathbf{e}}_{22}+\frac{v_3 v_2}{h_3}\hat{\mathbf{e}}_{23}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2+\frac{v_1 v_2}{h_1}\left(\frac{h_{12}}{h_2}\hat{\mathbf{e}}_1\right)$$

$$+\frac{v_2 v_2}{h_2}\left(-\frac{h_{21}}{h_1}\hat{\mathbf{e}}_1-\frac{h_{23}}{h_3}\hat{\mathbf{e}}_3\right)+\frac{v_3 v_2}{h_3}\left(\frac{h_{32}}{h_2}\hat{\mathbf{e}}_3\right)$$

$$=\left(\frac{v_1 v_2 h_{12}}{h_1 h_2}-\frac{v_2 v_2 h_{21}}{h_2 h_1}\right)\hat{\mathbf{e}}_1+\left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2+\left(\frac{v_3 v_2 h_{32}}{h_2 h_3}-\frac{v_2 v_2 h_{23}}{h_2 h_3}\right)\hat{\mathbf{e}}_3$$

$$(\mathbf{V}\cdot\nabla)(v_3\hat{\mathbf{e}}_3)=$$

$$=\frac{v_1}{h_1}\frac{\partial(v_3\hat{\mathbf{e}}_3)}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial(v_3\hat{\mathbf{e}}_3)}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial(v_3\hat{\mathbf{e}}_3)}{\partial x_3}$$

$$=\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}\hat{\mathbf{e}}_3+\frac{v_1 v_3}{h_1}\frac{\partial\hat{\mathbf{e}}_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}\hat{\mathbf{e}}_3+\frac{v_2 v_3}{h_2}\frac{\partial\hat{\mathbf{e}}_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\hat{\mathbf{e}}_3+\frac{v_3 v_3}{h_3}\frac{\partial\hat{\mathbf{e}}_3}{\partial x_3}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3+\frac{v_1 v_3}{h_1}\hat{\mathbf{e}}_{31}+\frac{v_2 v_3}{h_2}\hat{\mathbf{e}}_{32}+\frac{v_3 v_3}{h_3}\hat{\mathbf{e}}_{33}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3+\frac{v_1 v_3}{h_1}\left(\frac{h_{13}}{h_3}\hat{\mathbf{e}}_1\right)+\frac{v_2 v_3}{h_2}\left(\frac{h_{23}}{h_3}\hat{\mathbf{e}}_2\right)$$

$$+\frac{v_3 v_3}{h_3}\left(-\frac{h_{31}}{h_1}\hat{\mathbf{e}}_1-\frac{h_{32}}{h_2}\hat{\mathbf{e}}_2\right)$$

$$=\left(\frac{v_1 v_3 h_{13}}{h_1 h_3}-\frac{v_3 v_3 h_{31}}{h_3 h_1}\right)\hat{\mathbf{e}}_1+\left(\frac{v_2 v_3 h_{23}}{h_2 h_3}-\frac{v_3 v_3 h_{32}}{h_3 h_2}\right)\hat{\mathbf{e}}_2+\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3$$

Pressure gradient $\nabla p=\frac{1}{h_1}\frac{\partial p}{\partial x_1}\hat{\mathbf{e}}_1+\frac{1}{h_2}\frac{\partial p}{\partial x_2}\hat{\mathbf{e}}_2+\frac{1}{h_3}\frac{\partial p}{\partial x_3}\hat{\mathbf{e}}_3$

Viscous term $\nabla^2\mathbf{V}$

$$\begin{aligned}
&\nabla^2\mathbf{V}=\\
&=\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_1\\
&-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)\right]\hat{\mathbf{e}}_1\\
&+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_2\\
&-\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)\right]\hat{\mathbf{e}}_2\\
&+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_3\\
&-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)\right]\hat{\mathbf{e}}_3
\end{aligned}$$

5.2.1 Combine terms in $\hat{\mathbf{e}}_1$ direction to get momentum equation in $\hat{\mathbf{e}}_1$ direction

$$\begin{aligned}
&\frac{\partial v_1}{\partial t}+\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}+\frac{v_1v_2h_{12}}{h_1h_2}-\frac{v_2v_2h_{21}}{h_2h_1}+\frac{v_1v_3h_{13}}{h_1h_3}-\frac{v_3v_3h_{31}}{h_3h_1}\\
&\quad=-\frac{1}{\rho}\frac{1}{h_1}\frac{\partial p}{\partial x_1}\\
&+\nu\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\\
&-\nu\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)\right]
\end{aligned}$$

5.2.2 Combine terms in $\hat{\mathbf{e}}_2$ direction to get momentum equation in $\hat{\mathbf{e}}_2$ direction

$$\frac{\partial v_2}{\partial t}+\frac{v_2 v_1 h_{21}}{h_1 h_2}-\frac{v_1 v_1 h_{12}}{h_1 h_2}+\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}+\frac{v_2 v_3 h_{23}}{h_2 h_3}-\frac{v_3 v_3 h_{32}}{h_3 h_2}$$

$$=-\frac{1}{\rho}\frac{1}{h_2}\frac{\partial p}{\partial x_2}$$

$$+\nu\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(h_2 h_3 v_1\right)+\frac{\partial}{\partial x_2}\left(h_3 h_1 v_2\right)+\frac{\partial}{\partial x_3}\left(h_1 h_2 v_3\right)\right]\right]$$

$$-\nu\frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}\left(h_3 v_3\right)-\frac{\partial}{\partial x_3}\left(h_2 v_2\right)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(h_2 v_2\right)-\frac{\partial}{\partial x_2}\left(h_1 v_1\right)\right]\right)\right]$$

5.2.3 Combine terms in $\hat{\mathbf{e}}_3$ direction to get momentum equation in $\hat{\mathbf{e}}_3$ direction

$$\frac{\partial v_3}{\partial t}+\frac{v_3 v_1 h_{31}}{h_3 h_1}-\frac{v_1 v_1 h_{13}}{h_3 h_1}+\frac{v_3 v_2 h_{32}}{h_2 h_3}-\frac{v_2 v_2 h_{23}}{h_2 h_3}+\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}$$

$$=-\frac{1}{\rho}\frac{1}{h_3}\frac{\partial p}{\partial x_3}$$

$$+\nu\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(h_2 h_3 v_1\right)+\frac{\partial}{\partial x_2}\left(h_3 h_1 v_2\right)+\frac{\partial}{\partial x_3}\left(h_1 h_2 v_3\right)\right]\right]$$

$$-\nu\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\left[\frac{\partial}{\partial x_3}\left(h_1 v_1\right)-\frac{\partial}{\partial x_1}\left(h_3 v_3\right)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}\left(h_3 v_3\right)-\frac{\partial}{\partial x_3}\left(h_2 v_2\right)\right]\right)\right]$$

Rates of Strain and Rotation in Orthogonal Curvilinear Coordinates

A fluid element can undergo four different types of motion/deformation:

1. translation $\underline{V} = u\hat{i} + v\hat{j} + w\hat{k}$
2. rotation ω_{ij} = rigid body angular velocity
3. extensional strain ε_{ii} } rate of strain
4. shear strain ε_{ij} $i \neq j$ }

where ω_{ij} and ε_{ij} are the symmetric and anti symmetric parts of the deformation rate ie velocity gradient tensor

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij} \qquad \varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}(e_{ij} + e_{ji})$$

$$\omega_{ij} = -\omega_{ji} = \frac{1}{2}(e_{ij} - e_{ji})$$

which can also be written in vector form

$$e_{ij} = \nabla \underline{V}$$

$$\varepsilon_{ij} = \frac{1}{2}(\nabla \underline{V} + \nabla \underline{V}^T)$$

$$\omega_{ij} = \frac{1}{2}(\nabla \underline{V} - \nabla \underline{V}^T)$$

$e_{ij} = \frac{\partial u_i}{\partial x_j}$ can be derived both geometrically by analyzing the deformation of a fluid particle or mathematically by considering the relative motion between two neighboring fluid particles. The latter approach is more useful for extensions to curvilinear coordinate systems.

1. Geometric derivation of e_{ij}

Square element at t undergoes $\underline{V}$ and ω_{ij} and ε_{ij} and becomes of rhombic shape

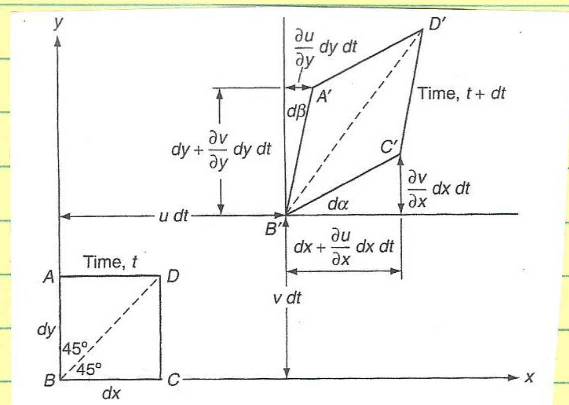


FIGURE 1-14
Distortion of a moving fluid element.

1. Translation

$$\underline{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt} \quad w = \frac{dz}{dt}$$

2. Rotation

$d\Omega_z = \frac{1}{2}(d\alpha - d\beta)$ = average counterclockwise rotation BC ($d\alpha$) and BA ($-d\beta$) about z axis

$$d\alpha = \lim_{dt\to 0} \tan^{-1} \frac{v_x\, dx\, dt}{dx + u_x\, dx\, dt} = v_x\, dt \qquad v_x = \frac{\partial v}{\partial x}$$

$$d\beta = \lim_{dt\to 0} \tan^{-1} \frac{u_y\, dy\, dt}{dy + v_y\, dy\, dt} = u_y\, dt \qquad u_y = \frac{\partial u}{\partial y}$$

$$\frac{d\Omega_z}{dt} = \frac{1}{2}(v_x - u_y) \quad \text{z axis}$$

$$\frac{d\Omega_x}{dt} = \frac{1}{2}(w_y - v_z) \quad \text{x axis}$$

$$\frac{d\Omega_y}{dt} = \frac{1}{2}(u_z - w_x) \quad \text{y axis}$$

} subscript denotes derivative for $\underline{V}$

$$\frac{d\underline{r}}{dt} = \frac{dr_x}{dt}\hat{i} + \frac{dr_y}{dt}\hat{j} + \frac{dr_z}{dt}\hat{k}$$

$$\underline{\omega} = 2\frac{d\underline{r}}{dt} = \nabla \times \underline{V} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

Subscript denotes angular velocity of vorticity components

note: $\nabla \cdot \underline{\omega} = 0$ since $\nabla \cdot \nabla \times \underline{V} = 0$

3. Shear strain = average decrease of angle between two lines that are initially ⊥ before strain

AB ⊥ BC at t

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right) = \frac{1}{2}(v_x + u_y)$$

$$\varepsilon_{yz} = \frac{1}{2}(w_y + v_z) \quad \wedge \quad \varepsilon_{zx} = \frac{1}{2}(u_z + w_x)$$

note: $\varepsilon_{ij} = \varepsilon_{ji} \quad i \neq j$

4. Extensional Strain = increase in length of fluid element

$$\varepsilon_{xx} = \frac{dx + u_x dx dt - dx}{dx} = u_x dt$$

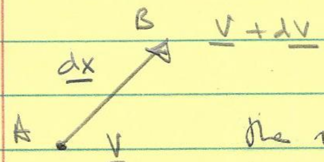
$$= u_x \qquad \varepsilon_{yy} = v_y \qquad \varepsilon_{zz} = w_z$$

$$\omega_{ij} = \begin{bmatrix} 0 & -\zeta & \eta \\ \zeta & 0 & -\xi \\ -\eta & \xi & 0 \end{bmatrix} = -\omega_{ji} \qquad \underline{\omega} = 2\xi\hat{i} + 2\eta\hat{j} + 2\zeta\hat{k}$$

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \varepsilon_{ji}$$

$$e_{ij} = \frac{\partial u_i}{\partial j} = \varepsilon_{ij} + \omega_{ij} = \tfrac{1}{2}\left(e_{ij} + e_{ji}\right) + \tfrac{1}{2}\left(e_{ij} - e_{ji}\right)$$

2. Relative Motion derivation e_{ij}



The velocity at B is estimated using a 1st order Taylor series

$$u_B = u_A + u_x dx + u_y dy + u_z dz + u_{xx}\frac{dx^2}{2} + \cdots$$

$$v_B = v_A + v_x dx + v_y dy + v_z dz + v_{xx}\frac{dx^2}{2} + \cdots$$

$$w_B = w_A + w_x dx + w_y dy + w_z dz + w_{xx}\frac{dx^2}{2} + \cdots$$

$$\underline{V_B} = \underline{V_A} + \underline{dV}$$

relative motion

$$\underline{dV} = \nabla_j u_i\, dx_j = dx_i \nabla_i u_j = e_{ij}\, dx_j = dx_i\, e_{ji}$$

$$= \nabla\underline{V} \cdot \underline{dx} = \underline{dx}\ \nabla\underline{V}^T$$

$$\nabla_j u_i = \begin{matrix} u_x & u_y & u_z & dx \\ v_x & v_y & v_z & dy \\ w_x & w_y & w_z & dz \end{matrix}$$

$$dx\ dy\ dz \quad \nabla_i u_j \quad \begin{matrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{matrix}$$

$$dx_i \underbrace{\nabla_i u_j}_{\nabla \underline{V}^T} = \underbrace{\nabla_j u_i}_{\nabla \underline{V}} dx_j$$

$\nabla \underline{V}$ = velocity gradient ie deformation rate tensor

in continuum mechanics called displacement gradient tensor, which is fundamental in the theory of elasticity

$$e_{ij} = \nabla \underline{V} \qquad e_{ji} = \nabla \underline{V}^T = \underline{V} \nabla$$

$$\varepsilon_{ij} = \tfrac{1}{2}(\nabla \underline{V} + \nabla \underline{V}^T) \qquad \omega_{ij} = \tfrac{1}{2}(\nabla \underline{V} - \nabla \underline{V}^T)$$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

For curvilinear case must be taken as u_i is a vector, ie

$$\nabla \underline{v} = \nabla_j \underline{u_i} = \nabla_j u_i \hat{e}_i = \frac{\partial u_i}{\partial x_j}$$

$$= \nabla_j u_i + u_i \nabla_j \hat{e}_i = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}$$

for Cartesian coordinates $\nabla_j \hat{e}_i = 0$, but more generally for curvilinear coordinates the derivatives of $\hat{e}_i$ must be included, e.g., the curvilinear coordinates

$$\nabla \underline{v} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} \qquad \nabla \underline{v}^T = \begin{bmatrix} a & d & h \\ b & e & i \\ c & f & j \end{bmatrix}$$

$$\varepsilon_{ij} = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T) = \frac{1}{2}\begin{bmatrix} 2a & b+d & c+h \\ d+b & 2e & f+i \\ h+c & i+f & 2j \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}(\nabla \underline{v} - \nabla \underline{v}^T) = \frac{1}{2}\begin{bmatrix} 0 & b-d & c-h \\ d-b & 0 & f-i \\ h-c & i-f & 0 \end{bmatrix}$$

Cartesian coordinates $a = u_x \quad b = u_y \quad c = u_z$

$d = v_x \quad e = v_y \quad f = v_z$

$h = w_x \quad i = w_y \quad j = w_z$

$$\varepsilon_{ij} = \frac{1}{2}\begin{bmatrix} 2u_x & u_y + v_x & u_z + w_x \\ v_x + u_y & 2v_y & v_z + w_y \\ w_x + u_z & w_y + v_z & 2w_z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a' & f' & g' \\ f' & b' & h' \\ g' & h' & c' \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2}\begin{bmatrix} 0 & u_y - v_x & u_z - w_x \\ v_x - u_y & 0 & v_z - w_y \\ w_x - u_z & w_y - v_z & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & f'' & g'' \\ -f'' & 0 & h'' \\ -g'' & -h'' & 0 \end{bmatrix}$$

$a' = 2a \qquad f' = b + d \qquad f'' = b - d$

$b' = 2e \qquad g' = c + h \qquad g'' = c - h$

$c' = 2j \qquad h' = f + i \qquad h'' = f - i$

Orthogonal Curvilinear Coordinates

$$\underline{V} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$$

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{e}_3$$

$$\nabla \underline{V}^T = \left[\frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{e}_3\right]^T \left[v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3\right]$$

$$= \frac{1}{h_1}\frac{\partial}{\partial x_1}\left(v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3\right)\hat{e}_1$$

$$+ \frac{1}{h_2}\frac{\partial}{\partial x_2}\left(\qquad \right)\hat{e}_2$$

$$+ \frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\qquad \right)\hat{e}_3$$

$$\hat{e}_1 \quad \frac{1}{h_1}\left[\frac{\partial}{\partial x_1}(v_1\hat{e}_1) + \frac{\partial}{\partial x_1}(v_2\hat{e}_2) + \frac{\partial}{\partial x_1}(v_3\hat{e}_3)\right]$$

$$\hat{e}_2 \quad \frac{1}{h_2}\left[\frac{\partial}{\partial x_2}(v_1\hat{e}_1) + \frac{\partial}{\partial x_2}(v_2\hat{e}_2) + \frac{\partial}{\partial x_2}(v_3\hat{e}_3)\right]$$

$$\hat{e}_3 \quad \frac{1}{h_3}\left[\frac{\partial}{\partial x_3}(v_1\hat{e}_1) + \frac{\partial}{\partial x_3}(v_2\hat{e}_2) + \frac{\partial}{\partial x_3}(v_3\hat{e}_3)\right]$$

$$\hat{e}_1 \quad \frac{1}{h_1}\left[\frac{\partial v_1}{\partial x_1}\hat{e}_1 + v_1\hat{e}_{11} + \frac{\partial v_2}{\partial x_1}\hat{e}_2 + v_2\hat{e}_{21} + \frac{\partial v_3}{\partial x_1}\hat{e}_3 + v_3\hat{e}_{31}\right]$$

$$= h_1\left[\frac{\partial v_1}{\partial x_1}\hat{e}_1 + v_1\left(-\frac{h_{12}}{h_2}\hat{e}_2 - \frac{h_{13}}{h_3}\hat{e}_3\right) + \frac{\partial v_2}{\partial x_1}\hat{e}_2 + v_2\left(\frac{h_{12}}{h_2}\hat{e}_1\right) + \frac{\partial v_3}{\partial x_1}\hat{e}_3 + v_3\left(\frac{h_{13}}{h_3}\hat{e}_1\right)\right]$$

a

$$\hat{e}_1 \quad \frac{1}{h_1}\left[\left(\frac{\partial v_1}{\partial x_1} + v_2\frac{h_{12}}{h_2} + v_3\frac{h_{13}}{h_3}\right)\hat{e}_1\right.$$

d

$$+\left(-v_1\frac{h_{12}}{h_2} + \frac{\partial v_2}{\partial x_1}\right)\hat{e}_2$$

h

$$\left. + \left(-v_1\frac{h_{13}}{h_3} + \frac{\partial v_3}{\partial x_1}\right)\hat{e}_3\right]$$

$\hat{e}_2 \quad \frac{1}{h_2}\left[\frac{\partial v_1}{\partial x_2}\hat{e}_1 + v_1\hat{e}_{12} + \frac{\partial v_2}{\partial x_2}\hat{e}_2 + v_2\hat{e}_{22} + \frac{\partial v_3}{\partial x_2}\hat{e}_3 + v_3\hat{e}_{32}\right]$

$$\frac{1}{h_2}\left[\frac{\partial v_1}{\partial x_2}\hat{e}_1 + v_1\left(\frac{h_{21}}{h_1}\right)\hat{e}_2 + \frac{\partial v_2}{\partial x_2}\hat{e}_2 + v_2\left(-\frac{h_{21}}{h_1}\hat{e}_1 - \frac{h_{23}}{h_3}\hat{e}_3\right) + \frac{\partial v_3}{\partial x_2}\hat{e}_3 + v_3\left(\frac{h_{23}}{h_3}\right)\hat{e}_2\right]$$

b, e

$\hat{e}_2 \quad \frac{1}{h_2}\left[\left(\frac{\partial v_1}{\partial x_2} - v_2\frac{h_{21}}{h_1}\right)\hat{e}_1 + \left(v_1\frac{h_{21}}{h_1} + \frac{\partial v_2}{\partial x_2} + v_3\frac{h_{23}}{h_3}\right)\hat{e}_2 + \left(-v_2\frac{h_{23}}{h_3} + \frac{\partial v_3}{\partial x_2}\right)\hat{e}_3\right]$

i

$\hat{e}_3 \quad \frac{1}{h_3}\left[\frac{\partial v_1}{\partial x_3}\hat{e}_1 + v_1\hat{e}_{13} + \frac{\partial v_2}{\partial x_3}\hat{e}_2 + v_2\hat{e}_{23} + \frac{\partial v_3}{\partial x_3}\hat{e}_3 + v_3\hat{e}_{33}\right]$

$$\frac{1}{h_3}\left[\frac{\partial v_1}{\partial x_3}\hat{e}_1 + v_1\left(\frac{h_{31}}{h_1}\right)\hat{e}_3 + \frac{\partial v_2}{\partial x_3}\hat{e}_2 + v_2\left(\frac{h_{32}}{h_1}\right)\hat{e}_3 + \frac{\partial v_3}{\partial x_3}\hat{e}_3 + \frac{\partial v_3}{\partial x_3}\left(-\frac{h_{31}}{h_1}\hat{e}_1 - \frac{h_{32}}{h_2}\hat{e}_2\right)\right]$$

c, f

$\hat{e}_3 \quad \frac{1}{h_3}\left[\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_3}\frac{h_{31}}{h_1}\right)\hat{e}_1 + \left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_3}\frac{h_{32}}{h_2}\right)\hat{e}_2 + \left(v_1\frac{h_{31}}{h_1} + v_2\frac{h_{32}}{h_1} + \frac{\partial v_3}{\partial x_3}\right)\hat{e}_3\right]$

j

$$a' = \frac{2}{h_1}\left(\frac{\partial v_1}{\partial x_1} + v_2 \frac{h_{12}}{h_2} + v_3 \frac{h_{13}}{h_3}\right) = 2a$$

$$b' = \frac{2}{h_2}\left(\frac{\partial v_2}{\partial x_2} + v_1 \frac{h_{21}}{h_1} + v_3 \frac{h_{23}}{h_3}\right) = 2e$$

$$c' = \frac{2}{h_3}\left(\frac{\partial v_3}{\partial x_3} + v_1 \frac{h_{31}}{h_1} + v_2 \frac{h_{32}}{h_1}\right) = 2j$$

xy $$f' = \underbrace{\frac{1}{h_1}\left(\frac{\partial v_2}{\partial x_1} - v_1 \frac{h_{12}}{h_2}\right)}_{d} + \underbrace{\frac{1}{h_2}\left(\frac{\partial v_1}{\partial x_2} - v_2 \frac{h_{21}}{h_1}\right)}_{b} = d + b = b + d$$

xz $$g' = \underbrace{\frac{1}{h_1}\left(\frac{\partial v_3}{\partial x_1} - v_1 \frac{h_{13}}{h_3}\right)}_{h} + \underbrace{\frac{1}{h_3}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_3} \frac{h_{31}}{h_1}\right)}_{c} = h + c = c + h$$

yz $$h' = \underbrace{\frac{1}{h_2}\left(-v_2 \frac{h_{23}}{h_3} + \frac{\partial v_3}{\partial x_2}\right)}_{i} + \underbrace{\frac{1}{h_3}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_3} \frac{h_{32}}{h_2}\right)}_{f} = i + f = f + i$$

$$f' = \frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{v_1}{h_1}\right) + \frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{v_2}{h_2}\right)$$

$$= \frac{h_1}{h_2 h_1}\frac{\partial v_1}{\partial x_2} - \frac{h_1}{h_2 h_1^2} h_{12} v_1 + \frac{h_2}{h_1 h_2}\frac{\partial v_2}{\partial x_1} - \frac{h_2}{h_1 h_2^2} h_{21} v_2$$

$$= \frac{1}{h_1}\left(\frac{\partial v_2}{\partial x_1} - \frac{h_{12}}{h_2} v_1\right) + \frac{1}{h_2}\left(\frac{\partial v_1}{\partial x_2} - \frac{h_{21}}{h_1} v_2\right)$$

$$g' = \frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{v_3}{h_3}\right) + \frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{v_1}{h_1}\right)$$

$$h' = \frac{h_2}{h_3}\frac{\partial}{\partial z}\left(\frac{v_2}{h_2}\right) + \frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{v_3}{h_3}\right)$$

$f'' = b - d \qquad -\omega_z$

$g'' = c - h \qquad \omega_\theta$

$h'' = f - i \qquad -\omega_r$

Cylindrical Coordinates

$$(x_1, y_1, z_1) = (r, \theta, z)$$

$$h_1 = h_r = 1 \quad h_2 = h_\theta = r \quad h_3 = h_z = 1$$

$$h_{21} = h_{\theta r} = 1 \quad h_{ij} \; {\scriptstyle i \neq 2 \atop j \neq 1} = 0$$

$$\varepsilon_{11} = \tfrac{1}{2} 2a = a = \frac{\partial v_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \tfrac{1}{2} 2e = e = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)$$

$$\varepsilon_{zz} = \tfrac{1}{2} 2j = j = \frac{\partial v_z}{\partial z}$$

$$\varepsilon_{r\theta} = \tfrac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right]$$

$$\varepsilon_{rz} = \tfrac{1}{2}\left[\frac{\partial v_z}{\partial x} + \frac{\partial v_r}{\partial z}\right]$$

$$\varepsilon_{\theta z} = \tfrac{1}{2}\left[\frac{1}{r}\left(\frac{\partial v_z}{\partial \theta}\right) + \frac{\partial v_\theta}{\partial z}\right]$$

$$f'' = b - d = \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) - \frac{\partial v_\theta}{\partial r} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \underbrace{\left(\frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right)}_{\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}}$$

$$g'' = c - h = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 2\omega_\theta$$

$$\left[\frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right]$$

$$h'' = \frac{\partial v_\theta}{\partial z} - \frac{1}{r}\frac{\partial v_z}{\partial \theta} = -2\omega_r \qquad = -2\omega_z$$

6. Example: Incompressible N-S equations in cylindrical polar systems

6.1 Continuity equation $\nabla \cdot \mathbf{V} = 0$ in cylindrical coordinates (r, θ, z)

For cylindrical coordinates (r, θ, z), $h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_z = 1$

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(rv_z)\right] = 0$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

6.2 Momentum equation $\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{V}$ in cylindrical coordinates (r, θ, z)

For cylindrical coordinates (r, θ, z), $h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_z = 1$ and only $h_{21} = h_{\theta r} = 1$, all others are zero.

6.2.1 The r-momentum equation:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{1}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

6.2.2 The θ-momentum equation:

$$\frac{\partial v_\theta}{\partial t} + \frac{1}{r}v_r v_\theta + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 F_2 - \frac{F_2}{r^2} + \frac{2}{r^2}\frac{\partial F_1}{\partial \theta}\right)$$

6.2.3 The z-momentum equation:

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 v_z$$

7. Overview extensions for nonorthogonal curvilinear coordinates

The Reynolds, continuity, turbulent-kinetic-energy k, and its dissipation-rate ε equations for steady incompressible flow can be written in the following vector form:

$$\frac{1}{2}\nabla(\underline{V}\cdot\underline{V}) - \underline{V}\times\underline{\omega} = -\nabla p/\rho + \nu\{\nabla(\nabla\cdot\underline{V}) - \nabla\times\underline{\omega}\}$$
$$- \nabla\cdot\overline{v_i v_j} + \overline{(v_i)\,\nabla\cdot\underline{v}} \qquad \text{(IV-1)}$$

$$\nabla\cdot\underline{V} = 0 \qquad \text{(IV-2)}$$

$$\underline{V}\cdot\nabla k = \nabla\cdot\left(\frac{\nu_t}{\sigma_k}\nabla k\right) + \tilde{G} - \varepsilon \qquad \text{(IV-3)}$$

$$\underline{V}\cdot\nabla\varepsilon = \nabla\cdot\left(\frac{\nu_t}{\sigma_\varepsilon}\nabla\varepsilon\right) + C_{\varepsilon 1}\tilde{G}\frac{\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \qquad \text{(IV-4)}$$

where $\underline{V} = (U,V,W)$ are the mean velocity components, $\underline{v} = (u,v,w)$ are the turbulent velocity components, p is the mean pressure, $\underline{\omega} = \nabla\times\underline{V}$ is the mean vorticity, $\overline{v_i v_j}$ are the Reynolds stresses (the overbar denotes time averaging), $k = \frac{1}{2}\overline{\underline{v}\cdot\underline{v}}$ is the turbulent kinetic energy, $\nu_t = C_\mu k^2/\varepsilon$ is the eddy viscosity, and $\tilde{G}$ is the turbulence generation. Since the fluid is assumed to be incompressible, the terms involving $\nabla\cdot\underline{V}$ and $\nabla\cdot\underline{v}$ in equation (IV-1) are identically zero, but have been included since they aid in putting the transformed equations into a more compact form. The usual values are used for the constants in the k-ε equations, namely, $(C_\mu, \sigma_k, \sigma_\varepsilon, C_{\varepsilon 1}, C_{\varepsilon 2}) = (.09, 1., 1.3, 1.44, 1.92)$. The turbulence generation term is defined by

$$\tilde{G} = \nu_t [2(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2) + 4(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2)] \qquad \text{(IV-5)}$$

where ε_{ij} is the rate-of-strain tensor

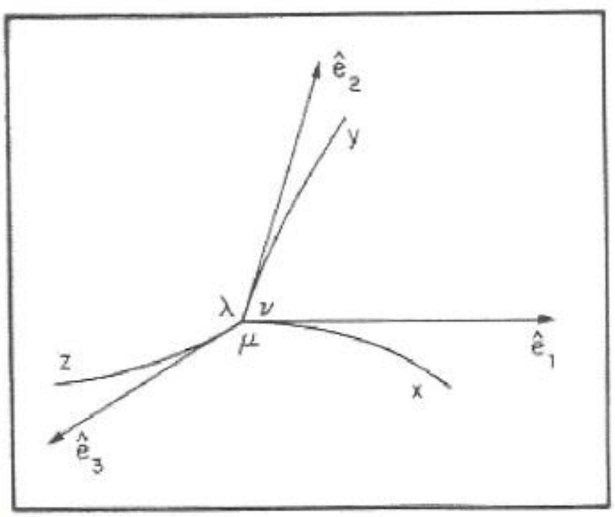
$$\varepsilon_{ij} = \frac{1}{2} [\nabla \underline{V} + \nabla \underline{V}^T] \qquad \text{(IV-6)}$$

In (IV-6) $\nabla \underline{V}$ is the deformation-rate tensor e_{ij} and $\nabla \underline{V}^T$ its transpose, i.e. $\nabla \underline{V}^T = e_{ji}$. The Reynolds stresses required in (IV-1) are related to k and ε through the isotropic eddy viscosity concept:

$$\overline{v_i v_j} = - 2\nu_t \varepsilon_{ij} + \frac{2}{3} k (h_i h_j g_{ij}) \qquad \text{(IV-7)}$$

where the h_i are the metric coefficients and g_{ij} is the inverse metric tensor both of which are defined below.

Equations (IV-1) - (IV-7) can be transformed into any coordinate system through the use of appropriate definitions of the gradient (∇), divergence ($\nabla\cdot$), and curl ($\nabla\times$) vector operators. The details of this procedure for orthogonal curvilinear coordinates are provided by Rouse (1959). For nonorthogonal curvilinear coordinates the appropriate vector operator definitions are not readily available. They were probably first derived by Weatherburn (1926). Following Weatherburn, and referring to figures 1 and 2 for the present notation, the unit vectors $\hat{e}_i = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ in the directions of the nonorthogonal curvilinear coordinates (x,y,z) are defined in terms of the body cartesian-coordinate position vector



Rouse, H. et al., (1976), "Advanced Mechanics of Fluids", Robert E. Krieger Publishing Company, Huntington, New York.

Weatherburn, C.E., (1926), "On Triple Systems of Surfaces and Nonorthogonal Curvilinear Coordinates", Proc. Roy. Soc., Edinburgh, Vol. 46.

$$\underline{R} = X\,(x,y,z)\,\hat{i} + Y\,(x,y,z)\,\hat{j} + Z\,(x,y,z)\,\hat{k} \qquad \text{(IV-8)}$$

by

$$\hat{e}_1 = \underline{R_x}/h_1,\ \hat{e}_2 = \underline{R_y}/h_2,\ \hat{e}_3 = \underline{R_z}/h_3 \qquad \text{(IV-9)}$$

where

$$h_1 = |\underline{R_x}|,\ h_2 = |\underline{R_y}|,\ h_3 = |\underline{R_z}| \qquad \text{(IV-10)}$$

and a lettered subscript denotes a partial derivative. The angles (λ, μ, ν) between the (x,y,z) coordinate axes are given by

$$\begin{aligned} \cos\lambda &= \hat{e}_2 \cdot \hat{e}_3 \\ \cos\mu &= \hat{e}_1 \cdot \hat{e}_3 \\ \cos\nu &= \hat{e}_1 \cdot \hat{e}_2 \end{aligned} \qquad \text{(IV-11)}$$

and the unit normals to constant x- y- and z-surfaces are given respectively by

$$\begin{aligned} \hat{e}_2 \times \hat{e}_3 &= \frac{1}{h_2 h_3 s}\,[Ah_1\hat{e}_1 + Hh_2\hat{e}_2 + Gh_3\hat{e}_3] \\ \hat{e}_3 \times \hat{e}_1 &= \frac{1}{h_1 h_3 s}\,[Hh_1\hat{e}_1 + Bh_2\hat{e}_2 + Fh_3\hat{e}_3] \\ \hat{e}_1 \times \hat{e}_2 &= \frac{1}{h_1 h_2 s}\,[Gh_1\hat{e}_1 + Fh_2\hat{e}_2 \quad Ch_3\hat{e}_3] \end{aligned} \qquad \text{(IV-12)}$$

where s is the triple product

$$s = (h_1 h_2 h_3)\ (\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3)$$

$$= [Ah_1^2 + Hh_1 h_2 \cos\nu + Gh_1 h_3 \cos\mu\,]^{1/2} \qquad \text{(IV-13)}$$

and

$$\begin{aligned}
A &= h_2^2 h_3^2 \sin^2\lambda \\
B &= h_1^2 h_3^2 \sin^2\mu \\
C &= h_1^2 h_2^2 \sin^2\nu \\
F &= (h_1 h_3 \cos\mu)\ (h_1 h_2 \cos\nu) - h_1^2\ (h_2 h_3 \cos\lambda) \\
G &= (h_1 h_2 \cos\nu)\ (h_2 h_3 \cos\lambda) - h_2^2\ (h_1 h_3 \cos\mu) \\
H &= (h_2 h_3 \cos\lambda)\ (h_1 h_3 \cos\mu) - h_3^2 (h_1 h_2 \cos\nu)
\end{aligned} \qquad \text{(IV-14)}$$

The inverse metric tensor is defined by

$$g_{ij} = (h_i \hat{e}_i \cdot h_j \hat{e}_j)^{-1}$$

$$= \frac{1}{s} \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix} \qquad \text{(IV-14.1)}$$

In terms of the above quantities, the gradient of any scalar $Q(x,y,z)$ and divergence and curl for any vector $\underline{V}(x,y,z) = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$ are given by:

$$\nabla Q = \frac{1}{s^2} \{ (AQ_x + HQ_y + GQ_z)\ h_1\hat{e}_1 + (HQ_x + BQ_y + FQ_z)\ h_2\hat{e}_2$$

$$+ (GQ_x + FQ_y + CQ_z)\, h_3\hat{e}_3\} \qquad \text{(IV-15)}$$

$$\nabla \cdot \underline{V} = \frac{1}{s}\left[\frac{\partial}{\partial x}\left(\frac{sV_1}{h_1}\right) + \frac{\partial}{\partial y}\left(\frac{sV_2}{h_2}\right) + \frac{\partial}{\partial z}\left(\frac{sV_3}{h_3}\right)\right] \qquad \text{(IV-16)}$$

$$\nabla x \underline{V} = \frac{1}{s}\{\frac{\partial}{\partial y}[V_1 h_3 \cos\mu + V_2 h_3 \cos\lambda + V_3 h_3]$$

$$- \frac{\partial}{\partial z}[V_1 h_2 \cos\nu + V_2 h_2 + V_3 h_2 \cos\lambda]\}\, h_1\hat{e}_1$$

$$+ \frac{1}{s}\{\frac{\partial}{\partial z}[V_1 h_1 + V_2 h_1 \cos\nu + V_3 h_1 \cos\mu]$$

$$- \frac{\partial}{\partial x}[V_1 h_3 \cos\mu + V_2 h_3 \cos\lambda + V_3 h_3]\}\, h_2\hat{e}_2$$

$$+ \frac{1}{s}\{\frac{\partial}{\partial x}[V_1 h_2 \cos\nu + V_2 h_2 + V_3 h_2 \cos\lambda]$$

$$- \frac{\partial}{\partial y}[V_1 h_1 + V_2 h_1 \cos\nu + V_3 h_1 \cos\mu]\}\, h_3\hat{e}_3 \qquad \text{(IV-17)}$$

The transformed equations are very lengthy and are provided in Appendix I. The equations have been put in a form that is similar to that used by Nash and Patel (1972) for orthogonal curvilinear coordinates. By comparison, it is seen that, for the present circumstances, the coefficients in the governing equations depend on terms related to not only the curvatures of the coordinates but also their angular orientation. Due to the complexity of the derivation of the transformed equations it was desired to check their accuracy; however, this was made difficult by the fact that no other presentations of the governing equations in nonorthogonal curvilinear coordinates in a format and notation similar to the present one are known to exist. The following checks were made: for $(\lambda, \mu, \nu) \to 90°$, the orthogonal form of the equations was recovered; for $(\lambda, \nu) \to 90°$, and subject to the boundary-layer assumptions the boundary-layer equations of Cebeci et al. (1978) were recovered; and some of the coefficients were compared with their corresponding counterparts in the tensor form of the equations presented by Richmond et al. (1986).