Chapters 1 Preliminary Concepts & 2 Fundamental Equations of Compressible Viscous Flow

(6.1) Reference Frames and Coordinate Systems

Reference frame → state of motion of the observer.

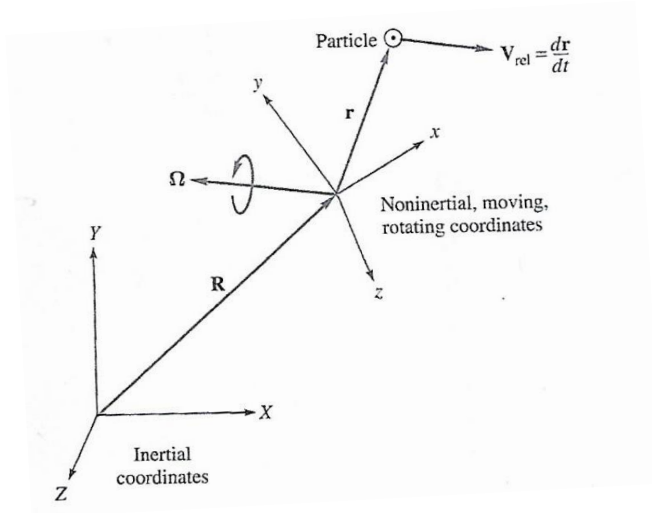
Coordinate system → set of numbers used to map the space points within a reference frame.

For any given reference frame, multiple coordinate systems are possible (e.g. Cartesian, spherical, etc.)

In classical physics and special relativity, an inertial frame of reference is a frame of reference that is not undergoing acceleration. In an inertial frame of reference, a physical object with zero net force acting on it moves with a constant velocity (which might be zero)—or, equivalently, it is a frame of reference in which Newton's first law of motion holds.

Non-inertial Reference Frame: A non-inertial reference frame is a frame of reference that undergoes acceleration with respect to an inertial frame.

Thus far we have assumed use of an inertial reference frame (i.e. fixed with respect to the distant stars in deriving the CV and differential form of the momentum equation). However, in many cases non-inertial reference frames are useful (e.g. rotational machinery, vehicle dynamics, geophysical applications, etc.).



Geometry of fixed (inertial) vs. accelerating (non-inertial) coordinates

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + \underline{a}_{rel}$$

$$\sum \underline{F} = m\underline{a}_i = m\left(\frac{D\underline{V}}{Dt} + \underline{a}_{rel}\right)$$

$$\sum \underline{F} - m\underline{a}_{rel} = m\frac{D\underline{V}}{Dt}$$

i.e Newton's law applies to non-inertial frame with addition of known inertial force terms

$\underline{V}$ = relative velocity in non-inertial reference frame (x,y,z)

$\frac{D\underline{V}}{Dt}$ = non-inertial $\underline{a}$ that must be added to $\underline{a}_{rel}$

The absolute fluid particle position vector is

$\underline{S}_i = \underline{R} + \underline{r}$

Such that

3rd term from fact that (x,y,z) rotating at Ω(t).

$$V_i = \underline{V} + \frac{d\underline{R}}{dt} + \underline{\Omega} \times \underline{r}$$

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + \frac{d^2\underline{R}}{dt^2} + \frac{d\underline{\Omega}}{dt} \times \underline{r} + 2\underline{\Omega} \times \underline{V} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r})$$

$$= \frac{D\underline{V}}{Dt} + \underline{a}_{rel}$$

$$\frac{d^2 \underline{R}}{dt^2} \qquad = \textit{acceleration } (x,y,z)$$

Bodies in non-inertial reference frames are subject to so-called fictitious forces (pseudo-forces); that is, forces that result from the acceleration of the reference frame itself and not from any physical force acting on the body. Examples of fictitious forces are the centrifugal force and the Coriolis force in rotating reference frames.

$-m\frac{d\underline{\Omega}}{dt}\times\underline{r}$ = Euler force due to *angular acceleration (x,y,z)*

$-m2\underline{\Omega}\times\underline{V}$ = *Coriolis force*

$-m\underline{\Omega}\times(\underline{\Omega}\times\underline{r})$ = *centrifugal force (directed away from the particle normal distance to the axis of rotation with magnitude -*$\Omega^2 L$*, where L = normal distance from* $\underline{r}$ *to axis of rotation* $\underline{\Omega}$*).*

Since $\underline{\mathrm{R}}$ and $\underline{\Omega}$ assumed known, although more complicated, we are simply adding known inhomogeneities to the momentum equation.

CV form of Momentum equation for non-inertial reference frame:

$$\sum \underline{F} - \int_{CV} \underline{a}_{rel}\rho \, d\forall = \frac{d}{dt}\int_{CV} \underline{V}\rho \, d\forall + \int_{CS} \underline{V}\rho\underline{V}_R \cdot \underline{n} \, dA$$

where $\underline{V}_R$ is the velocity of the CV relative to the non-inertial reference frame (x,y,z).

Differential form of momentum equation for non-inertial reference frame:

$$\rho\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = \underbrace{-\rho\underline{a}_{rel}}_{body\quad force} - \nabla\left(p+\gamma z\right) + \mu\nabla^2\underline{V}$$

where

$$\underline{a}_{rel} = \underline{\ddot{R}} + 2\underline{\Omega}\times\underline{V} + \underline{\Omega}\times\left(\underline{\Omega}\times\underline{r}\right) + \underline{\dot{\Omega}}\times\underline{r}$$

All terms in $\underline{a}_{rel}$ seldom act in unison (e.g. geophysical flows):

$\underline{\ddot{R}} \sim 0$ earth not accelerating relative to distant stars

$\underline{\dot{\Omega}} \sim 0$ for earth

$\underline{\Omega}\times\left(\underline{\Omega}\times\underline{r}\right) \sim 0$ g nearly constant with latitude

$\therefore 2\underline{\Omega}\times\underline{V}$ most important!

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + R_0^{-1}\left(2\underline{\Omega}\times\underline{V}\right) \qquad \underline{V} = \frac{\underline{V}}{V_0},\ t = \frac{tV_0}{L}$$

$$R_0 = Rossby\ \# = \frac{V_0^2/L}{\Omega V_0} = \frac{V_0}{\Omega L}$$

if L is large, i.e., comparable to the order of magnitude of the earth radius, $R_0<1$, then Coriolis term is larger than the inertia terms and is important.

Examples of Non-inertial Reference Frames

1. Geophysical fluids dynamics: Geophysics is a subject of natural science concerned with the physical processes and physical properties of the Earth and its surrounding space environment, and the use of quantitative methods for their analysis.

Atmosphere and oceans are naturally studied using non-inertial coordinate system rotating with the earth. Two primary forces are Coriolis force and buoyancy force due to density stratification ρ = ρ(T). Both are studied using Boussinesq approximations (ρ = constant, except $-\rho(T)g\hat{k}$ term; and μ, k, C_p = constant) and thin layer on rotating surface assumption $\left(\frac{W}{U} \sim \frac{H}{L}\right)$ where w/H are vertical, and U/L are horizontal scales.

Differences between atmosphere and oceans: lateral boundaries (continents) in oceans; currents in ocean (gulf and Kuroshio stream) along western boundaries; clouds and latent heat release in atmosphere due to moisture condensation; V_{ocean} = 0.1~1 or 2 m/s and $V_{atmosphere}$ 10~20 m/s

H << L ≈ 0 (radius of earth = 6371 km)
Therefore, one can neglect curvature of earth and replace spherical coordinates by local Cartesian tangent plane coordinates.

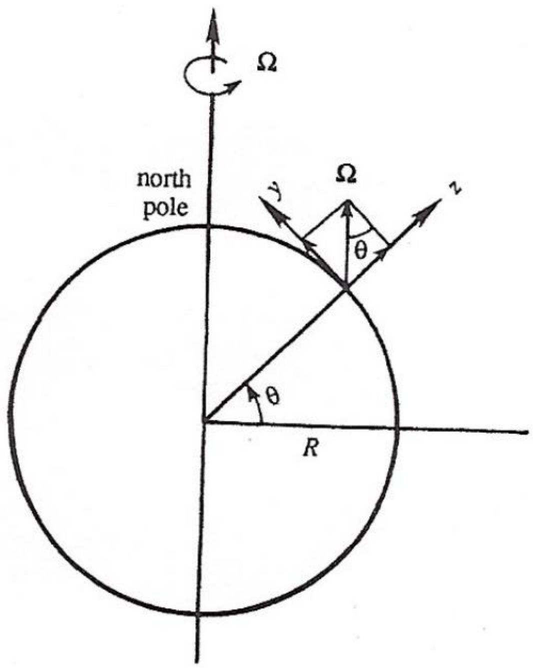


Fig. 13.3 Local Cartesian coordinates. The x-axis is into the plane of the paper.

x = eastward (into plane of paper)
y = northward tangent earth surface)
z = upward (opposing gravity)
$|\Omega| = \Omega = 2\pi$rad/day = $0.73\text{x}10^{-4}$ rad/s^{-1} around polar axis ccw above north pole

$\Omega_x = 0$

$\Omega_{y\,=}\ \Omega\cos\theta$

$\Omega_{z=}\ \Omega\sin\theta$

θ = latitude > 0 northern hemisphere and <0 southern hemisphere and = 0 at equator

Coriolis force = $2\underline{\Omega}\times\underline{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

0 since w << v

$$= 2\Omega\left[\hat{i}\left(w\cos\theta - v\sin\theta\right) + \hat{j}\,u\sin\theta - \hat{k}\,u\cos\theta\right]$$

$$= -fv\hat{i} + fu\hat{j} - 2\Omega\cos\theta\, u\,\hat{k} \qquad f = 2\Omega\sin\theta$$

= planetary vorticity
= 2 * vertical component Ω

$f > 0$	northern hemisphere
$f < 0$	southern hemisphere
$f = \pm\Omega$	at poles
$f = 0$	at equator

Person spins at Ω (at poles)

Person translates with inertial period $T_i = \dfrac{2\pi}{f}$ (at equator)

The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame and is proportional to the object's speed in the rotating frame (more precisely, to the component of its velocity that is perpendicular to the axis of rotation).

Equations of Motion

$$\nabla \cdot \underline{V} = 0$$

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \upsilon \nabla^2 u$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \upsilon \nabla^2 v$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} + \upsilon \nabla^2 w$$

$$\rho = \rho_0\left[1 - \alpha(T - T_0)\right]$$

vertical component Coriolis force negligible due to thin layer assumption, i.e., magnitude of $2\Omega\cos\theta u$ << *other terms*

p, ρ = perturbation from hydrostatic condition

Z momentum (assuming w = 0) → $\frac{\partial p}{\partial z} = -\rho g$ baroclinic fluid (such as the atmosphere) in which surfaces of constant pressure intersect those of constant density. p = p(T) since ρ = ρ(T) and can be used to eliminate p in above equations whereby (u,v) = f(T(z)), which is called thermal wind but not considered here.

Geostrophic Flow: quasi-steady, large-scale motions in atmosphere or ocean far from boundaries = wind/current due balance of grad(p) and Coriolis force

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad\qquad fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$\frac{D\underline{V}}{Dt} \sim 0\left(\frac{U^2}{L}\right) \quad f\underline{V} \sim 0\,(fU) \qquad$ U, L = horizontal scales

Rossby number = $\frac{U}{fL}$

Atmosphere: U ~ 10 m/s; f = 10^{-4} Hz; L ~ 1000 km; and $R_0 = 0.1$

Ocean: U ~ 0.1 m/s; f = 10^{-4} Hz; L ~ 1000 km; and $R_0 = 0.01$

Therefore, neglect $\frac{D\underline{V}}{Dt}$ and since there are no boundaries, neglect $\nu\nabla^2\underline{V}$.

If we neglect ρ=ρ(T) effects, (u,v) = f(p) and can be determined from measured p(x,y). Not valid near the equator ($\pm$ 3°) where f is small.

$$(u\hat{\imath} + v\hat{\jmath}) \cdot \nabla p = \frac{1}{\rho_0 f}\left(-\frac{\partial p}{\partial y}\hat{\imath} + \frac{\partial p}{\partial x}\hat{\jmath}\right) \cdot \left(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath}\right) = 0$$

i.e., $\underline{V}$ is perpendicular to ∇p → horizontal velocity is along (and not across) lines of constant horizontal pressure, which is reason isobars and streamlines coincide on a weather map! If f = constant, then $\psi = \frac{p}{\rho_0 f}$ can be regarded as a stream function $\psi_x = -v$ and $\psi_y = u$.

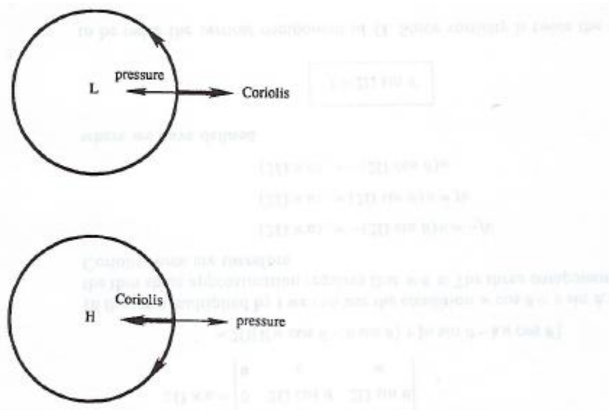
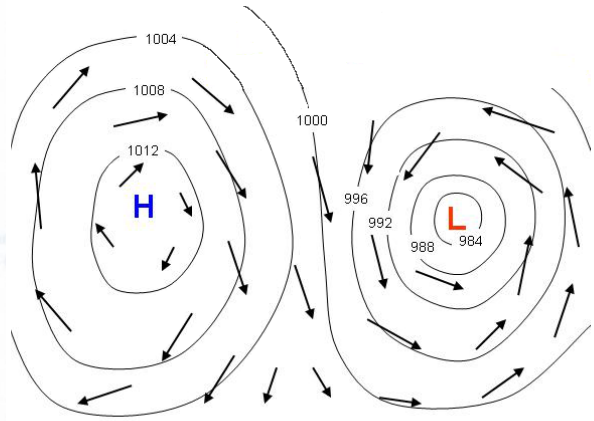


Fig. 13.4 Geostrophic flow around low and high pressure centers. The pressure force ($-\nabla p$) is indicated by a thin arrow, and the Coriolis force is indicated by a thick arrow.



Geostrophic flow around low- and high-pressure centers in the northern hemisphere. Coriolis force acts to the right of the velocity vector. Thus, the flow is counterclockwise (viewed form above) around low pressure and clockwise around high-pressure regions. The sense of circulation is opposite in the southern in the southern hemisphere where the Coriolis forces acts to the left of the velocity vector.

p=p(r) and $\nabla p = \frac{\partial p}{\partial r}\widehat{e_r}$

$\underline{V} = u_\theta(\text{r})\widehat{e_\theta}$

-$\rho 2\underline{\Omega} \times \underline{\text{V}}$=-$\nabla p$

$$\rho 2\Omega u_\theta \widehat{e_r} = -\frac{\partial p}{\partial r}\widehat{e_r}$$

$\frac{\partial p}{\partial r} < 0$ RHS +

$\frac{\partial p}{\partial r} > 0$ RHS -

Right hand rule: Coriolis force acts to the right of the velocity vector; therefore, when Coriolis force + u_θ CCW and when Coriolis force - u_θ CW.

2. Wind-Driven Flows: Impulsive start up and Ekman layers for free surface viscous flows.

Viscous layers:

Sudden acceleration flat plate: $u_t = \nu u_{yy}$ $\quad u(y,0) = 0$

$\delta = 3.64\sqrt{\nu t}$ $\quad u(0,t) = U$

$u(\infty,t) = 0$

Layer grows in time due viscous diffusion

Oscillating flat plate: $u_t = \upsilon u_{yy}$

$\delta = 6.5\sqrt{\nu / \omega}$ $\quad u(0,t) = U_0 \cos \omega t$

$u(\infty,t) = 0$

Layer confined constant thickness

Stagnation point flow: $\delta = 2.4\sqrt{\nu / B}$ and layer not a function of x since convection balances diffusion.

$$u_x + v_y = 0$$
$$uu_x + vu_y = -p_x + \upsilon(u_{xx} + u_{yy})$$
$$uv_x + vv_y = -p_y + \upsilon(v_{xx} + v_{yy})$$

u(x,o) = 0 and v(x,0) =0

Flat plate boundary layer:

$$u_x + v_y = 0$$
$$uu_x + vu_y = \upsilon u_{yy}$$
$$u(x,0) = 0$$
$$u(x,\infty) = U$$

$$\delta = 4.9\sqrt{\nu x/U}$$

Layer grows with $\sqrt{x}$ due convection.

Impulsive wind-driven free surface flow

Water at rest subject to impulsive wind stress τ_0. Assume unsteady parallel flow $u(z,t)$ and neglect coriolis force

$$u_t = \nu u_{zz}$$

$$\tau_0 = (\mu u_z)_{air} = (\mu u_z)_{water}\Big|_{z=0} \qquad \mu_{air} << \mu_{water}$$

$$u_z(0,t) = K = \tau_0/\mu_{water}$$

$$u(\infty,t) = 0$$

$$u(z,0) = 0$$

Similar Stokes 1st problem except the surface condition is for u_z vs u

$$u_z = K\left[1 + \text{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right]$$

$$u/K = z\left[1 + \text{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right] + 2\sqrt{\frac{\nu t}{\pi}}\, e^{-z^2/4\nu t}$$

$$u(0,t) = u_0 = 2K\sqrt{\frac{\nu t}{\pi}}$$

$$\tau_0 = .002\, \rho_{air}\, (V_{wind} - u_0)^2$$

For $V_{wind} = 6$ m/s (12 knots) and $T = 20°C \Rightarrow u_0 = .6$ m/s for $t = 1$ min and 2.3 m/s after 1 hour, which is greatly overestimated. Actual flow after 1 hour $u_0 = .2$ m/s or 3% V_{wind} due turbulence and need to use $\mu_t >> \mu$ in the solution

For Ekman layer viscous effects due to wind shear τ in x direction. Assume horizontal uniformity (i.e., $p_x = p_y = 0$), which is justified for L ~ 100 km and H ~ 50 m. The wind stress acts in the x direction.

$$-fv = \nu u_{ZZ} \qquad\qquad fu = \nu v_{ZZ}$$

$\mu u_z = \tau$ at z = 0

$v_z = 0$ at z = 0

$(u,v) = 0$ at z =-∞

Multiply v-equation by $i = \sqrt{-1}$ and add to u-equation:

$$\frac{d^2V}{dz^2} = \frac{i\,f}{\nu}V \qquad \begin{aligned} V &= u + i\,v \\ &= complex\ velocity \end{aligned}$$

$$V = Ae^{(1+i)z/\delta} + Be^{-(1+i)z/\delta}$$

$\delta = \sqrt{\dfrac{2\nu}{f}} =$ Ekman layer thickness

B = 0 for u(-∞), v(-∞) = 0

$\mu \dfrac{dV}{dz} = \tau$ at z = 0

$$\rightarrow A = \frac{\tau\delta(1-i)}{2\rho\nu}$$

i.e. $u = \frac{\tau/\rho}{\sqrt{f\upsilon}} e^{z/\delta} \cos\left(-\frac{z}{\delta}+\frac{\pi}{4}\right)$ and

$$v = \frac{\tau/\rho}{\sqrt{f\upsilon}} e^{z/\delta} \sin\left(-\frac{z}{\delta}+\frac{\pi}{4}\right)$$

$V_0 = \frac{\tau/\rho}{\sqrt{f\upsilon}}$ is the surface-water speed and $\delta = \sqrt{\frac{2\nu}{f}}$ is the penetration depth.

F. Nansen (1902) observed drifting arctic ice drifted 20-40^0 to the right of the wind, which he attributed to Coriolis acceleration. His student Ekman (1905) derived the solution. Recall f < 0 in southern hemisphere, so the drift is to the left of τ.

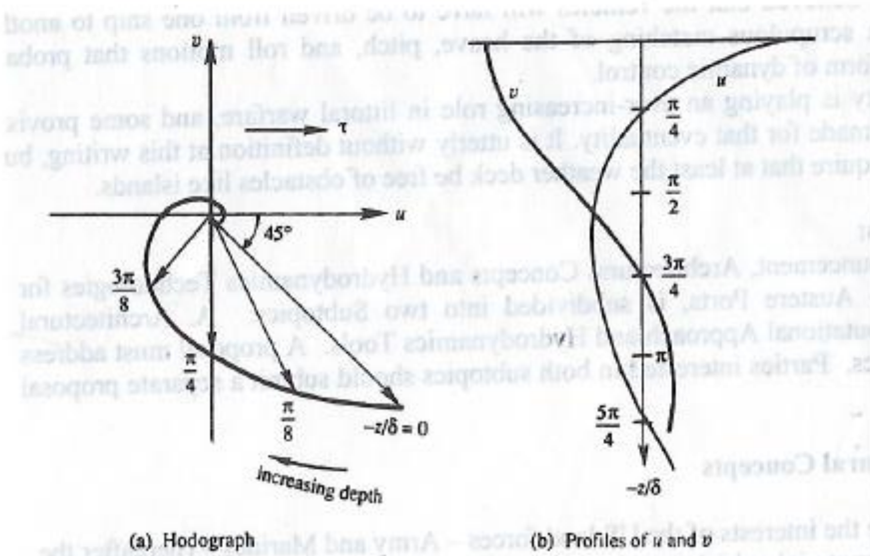


Figure 14.7 Ekman layer at a free surface. The left panel shows velocity at various depths; values of $-z/\delta$ are indicated along the curve traced out by the tip of the velocity vectors. The right panel shows vertical distributions of u and v.

The surface water velocities are

$$u = V_0 \cos\left(\frac{\pi}{4}\right)$$
$$v = V_0 \sin\left(\frac{\pi}{4}\right)$$

Showing that the surface velocity vector is 45 deg. angle to the right of the wind in the northern hemisphere. As we move below the surface the resultant current vector moves uniformly to the right and decreases exponentially in magnitude forming a logarithmic spiral. At z=-3δ/4 the current is exactly opposite to the wind and has magnitude $V_0 e^{-3\pi/4} \approx .095V_0$.

The components of the volume transport are:

$$\int_{-\infty}^{0} u\,dz = 0 = \frac{1}{\rho f}\int_{-\infty}^{0}\frac{\partial \tau_y}{\partial z}\,dz = \frac{1}{\rho f}\int_{-\infty}^{0} d\tau_y = \frac{1}{\rho f}(0-0) = 0$$

$$\int_{-\infty}^{0} v\,dz = -\frac{\tau}{\rho f} = \frac{1}{\rho f}\int_{-\infty}^{0}\frac{\partial \tau_x}{\partial z}\,dz = \frac{1}{\rho f}\int_{-\infty}^{0} d\tau_x = \frac{1}{\rho f}(0-\tau) = -\frac{\tau}{\rho f}$$

Which shows that the net transport is to the right of the applied stress and independent of ν due to the fact that the depth integrated Coriolis forces are directed to the right of the depth-integrated volume transport which balances the wind stress.

The horizontal uniformity can be removed easily if assume p is not function p(z) such that geostrophic solution is additive and combined solution recovers former for large depths $z/\delta \rightarrow -\infty$.

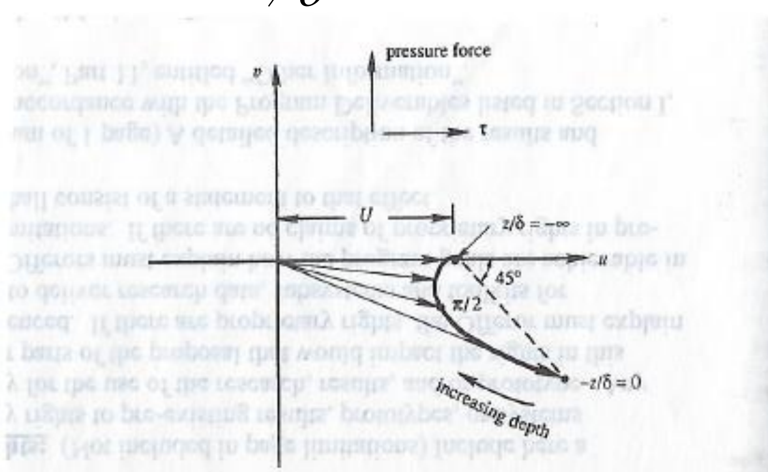


Figure 14.8 Ekman layer at a free surface in the presence of a pressure gradient. The geostrophic velocity forced by the pressure gradient is U.

For a Rhode Island Ekman layer V_{wind} = 6 m/s and T=20^o C with latitude $\theta = 41^0$ N: Laminar solution V_0 = 2.7 m/s and δ = 45 cm, which are too high/low; however, using turbulent ν_t, V_0 = 2 cm/s and δ = 100 m, which is more realistic. However, constant eddy viscosity and steady flow are unrealistic such that Ekman layers are not often observed.

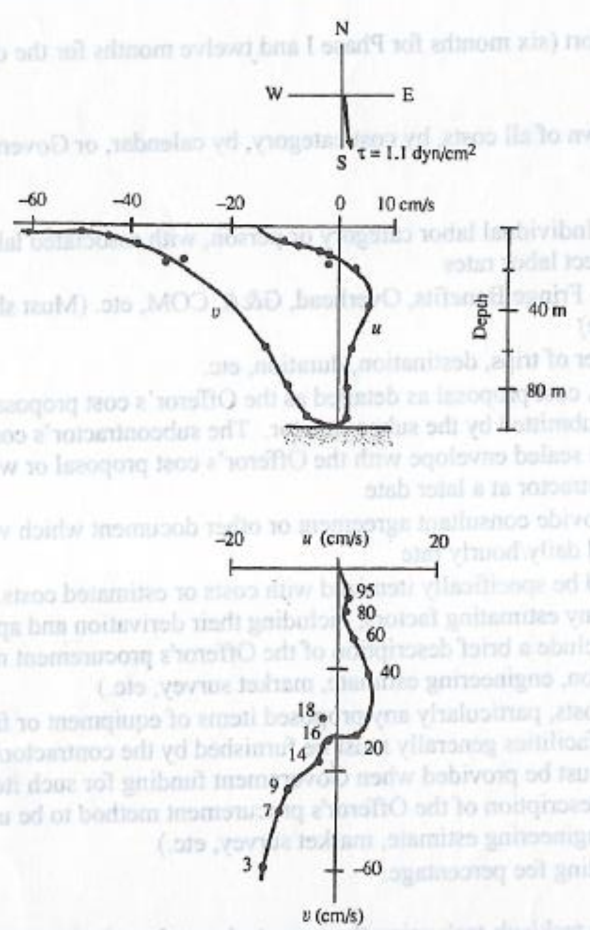


Figure 14.9 An observed velocity distribution near the coast of Oregon. Velocity is averaged over 7 days. Wind stress had a magnitude of 1.1 dyn/cm^2 and was directed nearly southward, as indicated at the top of the figure. The upper panel shows vertical distributions of *u* and *v*, and the lower panel shows the hodograph in which depths are indicated in meters. The hodograph is similar to that of a surface Ekman layer (of depth 16 m) lying over the bottom Ekman layer (extending from a depth of 16 m to the ocean bottom). P. Kundu, in *Bottom Tubulence*, J. C. J. Nihoul, ed., Elsevier, 1977 and reprinted with the permission of Jacques C. J. Nihoul.

The Ekman layer thickness is constant in time and space, which can be explained as follows

$$\omega_x = w_y - v_z = -v_z \qquad \text{since } w = 0$$

$$\omega_y = u_z - w_x = u_z$$

and

$$-f v_z = \nu \frac{d^2 (u_z)}{dz^2} = \nu \frac{d^2 \omega_y}{dz^2}$$

recall $-fv = \nu u_{zz}$

$fu = \nu v_{zz}$

$$-f u_z = \nu \frac{d^2 (v_z)}{dz^2} = \nu \frac{d^2 \omega_x}{dz^2}$$

which shows the vortex diffusion balances the Coriolis effects. The vertical fluid lines coincide with the planetary vortex lines. The tilting of the vertical fluid lines as per LHS causes a rate of change of ω_x and ω_y that cancels the viscous diffusion term.