Chapter 5: Energy Decay in Isotropic Turbulence

Part 3: Equation for Two-Point Correlations & Self-Preservation and the Karman-Howarth Equation

5.5 Equation for Two-Point Correlations

The analysis of isotropic decay carried out in the previous section concentrates on tracing the history of K and ϵ as they change in time. Only minimal information about the flow structure was needed, in fact, just the skewness and palenstrophy coefficient that are related to the two-point correlation functions. To proceed to a more extensive analysis of the decay problem that includes analyzing the time dependence of G^* and S_K^* it is necessary to include dynamical information about multi-point correlations. This means introducing an equation for the time history of the two-point velocity correlation tensor $\mathcal{R}(\mathbf{r}, t)$ and then considering its form during isotropic decay. From such an analysis it is also possible to consider the spectral properties of the turbulence during the decay process.

An equation governing $\mathcal{R}_{ij}(\mathbf{x}, \mathbf{y}, t)$ for arbitrary incompressible flow is derived by taking the average of $u_i(\mathbf{x}, t)$ times the jth component of the Navier–Stokes equation in Eq. (2.2) at $\mathbf{y}$ and adding to this the same quantity with i and j and $\mathbf{x}$ and $\mathbf{y}$ reversed. The result is

$$\begin{aligned}
&\overline{\rho u_i(\mathbf{x},t)\frac{\partial \mathcal{U}_j}{\partial t}(\mathbf{y},t)} + \overline{\rho u_j(\mathbf{y},t)\frac{\partial \mathcal{U}_i}{\partial t}(\mathbf{x},t)} \\
&+ \overline{\rho u_i(\mathbf{x},t)\mathcal{U}_k(\mathbf{y},t)\frac{\partial \mathcal{U}_j}{\partial y_k}(\mathbf{y},t)} + \overline{\rho u_j(\mathbf{y},t)\mathcal{U}_k(\mathbf{x},t)\frac{\partial \mathcal{U}_i}{\partial x_k}(\mathbf{x},t)} = \\
&- \overline{u_i(\mathbf{x},t)\frac{\partial p}{\partial y_j}(\mathbf{y},t)} - \overline{u_j(\mathbf{y},t)\frac{\partial p}{\partial x_i}(\mathbf{x},t)} \\
&+ \overline{\mu u_i(\mathbf{x},t)\nabla^2\mathcal{U}_j(\mathbf{y},t)} + \overline{\mu u_j(\mathbf{y},t)\nabla^2\mathcal{U}_i(\mathbf{x},t)}.
\end{aligned} \tag{5.83}$$

Using the definition of $\mathcal{R}_{ij}$ given in Eq. (2.30) it follows that the first two terms on the left-hand side of Eq. (5.83) may be written as

$$\overline{\rho u_i(\mathbf{x},t)\frac{\partial \mathcal{U}_j}{\partial t}(\mathbf{y},t)} + \overline{\rho u_j(\mathbf{y},t)\frac{\partial \mathcal{U}_i}{\partial t}(\mathbf{x},t)} = \rho\frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{x},\mathbf{y},t) \tag{5.84}$$

since terms such as $\overline{u_i(\mathbf{x},t)\partial\overline{\mathcal{U}}_j(\mathbf{y},t)/\partial t} \equiv 0$. The next two terms in Eq. (5.83), coming from the advection term, give

$$\begin{aligned}
&\overline{\rho u_i(\mathbf{x},t)\mathcal{U}_k(\mathbf{y},t)\frac{\partial \mathcal{U}_j}{\partial y_k}(\mathbf{y},t)} + \overline{\rho u_j(\mathbf{y},t)\mathcal{U}_k(\mathbf{x},t)\frac{\partial \mathcal{U}_i}{\partial x_k}(\mathbf{x},t)} = \\
&\overline{\rho u_i(\mathbf{x},t)u_k(\mathbf{y},t)}\frac{\partial \overline{\mathcal{U}}_j}{\partial y_k}(\mathbf{y},t) + \overline{\rho u_j(\mathbf{y},t)u_k(\mathbf{x},t)}\frac{\partial \overline{\mathcal{U}}_i}{\partial x_k}(\mathbf{x},t) + \\
&\rho\overline{\mathcal{U}}_k(\mathbf{y},t)\overline{u_i(\mathbf{x},t)\frac{\partial u_j}{\partial y_k}(\mathbf{y},t)} + \rho\overline{\mathcal{U}}_k(\mathbf{x},t)\overline{u_j(\mathbf{y},t)\frac{\partial u_i}{\partial x_k}(\mathbf{x},t)} + \\
&\overline{\rho u_i(\mathbf{x},t)u_k(\mathbf{y},t)\frac{\partial u_j}{\partial y_k}(\mathbf{y},t)} + \overline{\rho u_j(\mathbf{y},t)u_k(\mathbf{x},t)\frac{\partial u_i}{\partial x_k}(\mathbf{x},t)}.
\end{aligned} \tag{5.85}$$

The first two terms on the right-hand side of Eq. (5.85) are equal to

$$\rho\mathcal{R}_{ik}(\mathbf{x},\mathbf{y},t)\frac{\partial \overline{\mathcal{U}}_j}{\partial y_k}(\mathbf{y},t) + \rho\mathcal{R}_{jk}(\mathbf{y},\mathbf{x},t)\frac{\partial \overline{\mathcal{U}}_i}{\partial x_k}(\mathbf{x},t). \tag{5.86}$$

P5.6 Bernard

Furthermore, differentiation of Eq. (2.30) gives

$$\frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x},\mathbf{y},t) = \overline{u_i(\mathbf{x},t)\frac{\partial u_j}{\partial y_k}(\mathbf{y},t)} \tag{5.87}$$

and similarly for x_k derivatives, so that the third and fourth terms on the right-hand side of Eq. (5.85) take the form of convection terms

$$\rho\overline{U}_k(\mathbf{y},t)\frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x},\mathbf{y},t) + \rho\overline{U}_k(\mathbf{x},t)\frac{\partial \mathcal{R}_{ij}}{\partial x_k}(\mathbf{x},\mathbf{y},t). \tag{5.88}$$

As far as the last two terms on the right-hand side of Eq. (5.85) are concerned, they may be written using Eq. (2.31) as

$$\rho\overline{u_i(\mathbf{x},t)u_k(\mathbf{y},t)\frac{\partial u_j}{\partial y_k}(\mathbf{y},t)} = \rho\frac{\partial S_{jk,j}}{\partial y_k}(\mathbf{y},\mathbf{x},t) \tag{5.89}$$

and

$$\rho\overline{u_j(\mathbf{y},t)u_k(\mathbf{x},t)\frac{\partial u_i}{\partial x_k}(\mathbf{x},t)} = \rho\frac{\partial S_{ik,j}}{\partial x_k}(\mathbf{x},\mathbf{y},t) \tag{5.90}$$

where the fact that

$$\overline{u_i(\mathbf{x},t)\frac{\partial u_j}{\partial x_j}(\mathbf{x},t)u_k(\mathbf{y},t)} = 0 \tag{5.91}$$

has been used as implied by incompressibility.

To treat the contribution to Eq. (5.83) from the terms containing pressure, introduce the two-point pressure-velocity correlation vector

$$\mathcal{K}_i(\mathbf{x},\mathbf{y},t) = \overline{u_i(\mathbf{x},t)p(\mathbf{y},t)} \tag{5.92}$$

and see that

$$\overline{u_i(\mathbf{x},t)\frac{\partial p}{\partial y_j}(\mathbf{y},t)} + \overline{u_j(\mathbf{y},t)\frac{\partial p}{\partial x_i}(\mathbf{x},t)} = \frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x},\mathbf{y},t) + \frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y},\mathbf{x},t). \tag{5.93}$$

Finally, the viscous terms in Eq. (5.83) become

$$\mu\overline{u_i(\mathbf{x},t)\nabla^2 U_j(\mathbf{y},t)} + \mu\overline{u_j(\mathbf{y},t)\nabla^2 U_i(\mathbf{x},t)} = \mu\frac{\partial^2 \mathcal{R}_{ij}}{\partial y_k^2}(\mathbf{x},\mathbf{y},t) + \mu\frac{\partial^2 \mathcal{R}_{ij}}{\partial x_k^2}(\mathbf{x},\mathbf{y},t). \tag{5.94}$$

Putting the above results together it has been shown that Eq. (5.83) becomes

$$\begin{aligned}
&\frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{x},\mathbf{y},t) + \overline{U}_k(\mathbf{y},t)\frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x},\mathbf{y},t) + \overline{U}_k(\mathbf{x},t)\frac{\partial \mathcal{R}_{ij}}{\partial x_k}(\mathbf{x},\mathbf{y},t)\\
&= -\mathcal{R}_{ik}(\mathbf{x},\mathbf{y},t)\frac{\partial \overline{U}_j}{\partial y_k}(\mathbf{y},t) - \mathcal{R}_{jk}(\mathbf{y},\mathbf{x},t)\frac{\partial \overline{U}_i}{\partial x_k}(\mathbf{x},t)\\
&\quad - \frac{\partial S_{jk,i}}{\partial y_k}(\mathbf{y},\mathbf{x},t) - \frac{\partial S_{ik,j}}{\partial x_k}(\mathbf{x},\mathbf{y},t) - \frac{1}{\rho}\frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x},\mathbf{y},t)\\
&\quad - \frac{1}{\rho}\frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y},\mathbf{x},t) + \nu\frac{\partial^2 \mathcal{R}_{ij}}{\partial y_k^2}(\mathbf{x},\mathbf{y},t) + \nu\frac{\partial^2 \mathcal{R}_{ij}}{\partial x_k^2}(\mathbf{x},\mathbf{y},t).
\end{aligned}$$

Only hypothesis is incompressibility.

If $\underline{x} = \underline{y}$ recover Reynolds stress equation for incompressible flow

- Time derivative of $\mathcal{R}_{ij}$
- Convective transport of $\mathcal{R}_{ij}$
- Production terms
- Flux terms due to $S_{ik,j}$ related to vortex stretching.
- Flux terms due to two-point pressure-velocity correlation
- Viscous tensorial dissipation

When $\mathbf{x} = \mathbf{y}$, $\mathcal{R}_{ij}(\mathbf{x}, \mathbf{x}, t) = R_{ij}(\mathbf{x}, t)$, and it may be shown that Eq. (5.95) becomes identical to Eq. (3.53). This connection suggests that the first two terms on the right-hand side of Eq. (5.95) are "production" terms. The remaining terms acquire meaning by noting their similarity to the corresponding terms in Eq. (3.53).

The formidable complexity of Eq. (5.95) can be reduced somewhat by applying the relation to the specific case of homogeneous, isotropic turbulence. Since $\overline{U}_k(\mathbf{y}, t) = \overline{U}_k(\mathbf{x}, t)$ in homogeneous turbulence, and using results like Eq. (5.87), it follows that the two convection terms on the left-hand side of Eq. (5.95) sum to zero. Uniformity of $\overline{U}_i$ also implies that the two production terms on the right-hand side of Eq. (5.95) are zero.

The simplification for homogeneous turbulence used in Eq. (4.1) can be generalized to include the statements that

$$\mathcal{S}_{ij,k}(\mathbf{x}, \mathbf{y}, t) = \mathcal{S}_{ij,k}(\mathbf{y} - \mathbf{x}, t) \tag{5.96}$$

and

$$\mathcal{K}_i(\mathbf{x}, \mathbf{y}, t) = \mathcal{K}_i(\mathbf{y} - \mathbf{x}, t), \tag{5.97}$$

where for convenience the same symbols $\mathcal{R}_{ij}$, $\mathcal{S}_{ij,k}$, and $\mathcal{K}_i$ on the right-hand side are adopted; their applicability to homogeneous turbulence is implied by the appearance of one less argument than their more general counterparts. Using these relations, it follows that

$$\frac{\partial \mathcal{S}_{jk,i}}{\partial y_k}(\mathbf{y}, \mathbf{x}, t) = -\frac{\partial \mathcal{S}_{jk,i}}{\partial r_k}(\mathbf{x} - \mathbf{y}, t) \tag{5.98}$$

and

$$\frac{\partial \mathcal{S}_{ik,j}}{\partial x_k}(\mathbf{x}, \mathbf{y}, t) = -\frac{\partial \mathcal{S}_{ik,j}}{\partial r_k}(\mathbf{y} - \mathbf{x}, t), \tag{5.99}$$

and that

$$\frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x}, \mathbf{y}, t) = \frac{\partial \mathcal{K}_i}{\partial r_j}(\mathbf{y} - \mathbf{x}, t) \tag{5.100}$$

and

$$\frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y}, \mathbf{x}, t) = \frac{\partial \mathcal{K}_j}{\partial r_i}(\mathbf{x} - \mathbf{y}, t). \tag{5.101}$$

Putting together the various results, it is found that the two-point velocity correlation tensor in homogeneous turbulence is governed by the equation

$$\begin{aligned}\frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{r}, t) = {} & \frac{\partial \mathcal{S}_{jk,i}}{\partial r_k}(-\mathbf{r}, t) + \frac{\partial \mathcal{S}_{ik,j}}{\partial r_k}(\mathbf{r}, t) \\ & -\frac{1}{\rho}\frac{\partial \mathcal{K}_i}{\partial r_j}(\mathbf{r}, t) - \frac{1}{\rho}\frac{\partial \mathcal{K}_j}{\partial r_i}(-\mathbf{r}, t) + 2\nu\frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k^2}(\mathbf{r}, t).\end{aligned} \tag{5.102}$$

Contracting the indices in Eqs. (5.100) and (5.101), noting the definition of $\mathcal{K}_i$ in Eq. (5.92) and using the incompressibility condition gives in both cases

$$\frac{\partial \mathcal{K}_i}{\partial r_i}(\mathbf{r}, t) = 0. \tag{5.103}$$

Now taking a trace of Eq. (5.102) and using (5.103) gives

$$\frac{\partial \mathcal{R}_{ii}}{\partial t}(\mathbf{r},t) = \frac{\partial S_{ik,i}}{\partial r_k}(-\mathbf{r},t) + \frac{\partial S_{ik,i}}{\partial r_k}(\mathbf{r},t) + 2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\mathbf{r},t), \tag{5.104}$$

which shows that the time rate of change of the trace of the two-point velocity correlation tensor depends on a balance between viscous diffusion, given in the last term, and the two terms depending on the two-point triple velocity correlation tensor. The latter represent the process by which vortex stretching brings energy to small dissipative scales.

Note that the $\mathcal{R}_{ij}$ and $\mathcal{R}_{ii}$ equations are not closed as they contain the two-point triple velocity correlation terms and if equations derived for $S_{ik,i}$ they would contain fourth order velocity correlation terms, i.e., conundrum of the RANS turbulence closure problem and paradox.

Next impose isotropy by using isotropic tensor form of $\mathcal{R}_{ii}$ and $S_{ik,i}$ to obtain the K-H equation.

Self-Preservation and the Karman-Howarth Equation

The $\mathcal{R}_{ii}$ equation is transformed to the Karman-Howarth equation under the assumptions of homogeneous and isotopic turbulence.

$$\underbrace{\frac{\partial \mathcal{R}_{ii}}{\partial t}\left(\underline{r}, t\right)}_{\boxed{1}} - \underbrace{\frac{\partial S_{ik,i}}{\partial r_k}\left(-\underline{r}, t\right)}_{\boxed{2}} + \underbrace{\frac{\partial S_{ik,i}}{\partial r_k}\left(\underline{r}, t\right)}_{\boxed{3}} = \underbrace{2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}\left(\underline{r}, t\right)}_{\boxed{4}} \qquad (1)$$

Time derivative of a scalar	Divergence of a vector	Laplacian of a scalar

This is a scalar equation, where each term is function of $\underline{r}$ and t, in the most general case.

Combining Eq. (1) with the Chapter 4 Part 2 isotropic expressions for $\mathcal{R}_{ij}$ and $S_{ij,l}$

$$\mathcal{R}_{ij}\left(\underline{r}, t\right) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}\frac{df}{dr}\right]$$

$$S_{ijl}\left(\underline{r}, t\right) = u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_j r_l}{2r^3} - \frac{k}{2}\delta_{ij}\frac{r_l}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

it is possible to analyze each term in Eq. (1) separately.

Term 1:

$$\mathcal{R}_{ii}(\underline{r},t) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\underbrace{\delta_{ii}}_{\boxed{3}} - \frac{\cancel{r_i r_i}}{\cancel{r^2}}\frac{r}{2}\frac{df}{dr}\right] = \overline{u^2}\left(3f + r\frac{df}{dr}\right)$$

$$3f + r\frac{df}{dr} = \frac{1}{r^2}\left(3fr^2 + r^3\frac{df}{dr}\right) = \frac{1}{r^2}\left(\cancel{3fr^2} + \frac{d(r^3 f)}{dr} - \cancel{3fr^2}\right) = \frac{1}{r^2}\frac{d(r^3 f)}{dr}$$

$$\times\frac{r^2}{r^2}$$

$$\frac{\partial\mathcal{R}_{ii}}{\partial t}(\underline{r},t) = \frac{\partial}{\partial t}\left[\overline{u^2}\frac{1}{r^2}\frac{d(r^3 f)}{dr}\right]$$

Terms 2 and 3:

$$S_{ik,i}(\underline{r},t) = u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_k r_i}{2r^3} - \frac{k}{2}\delta_{ik}\frac{r_i}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{ii}\frac{r_k}{r} + \delta_{ki}\frac{r_i}{r}\right)\right]$$

$$= u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_k}{2r} - \frac{k}{2}\frac{r_k}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(3\frac{r_k}{r} + \frac{r_k}{r}\right)\right]$$

$$= u_{rms}^3\left[-\frac{dk}{dr}\frac{r_k}{2} + \frac{r_k}{r^2}\frac{d(kr^2)}{dr}\right]$$

$$= u_{rms}^3\left[-\frac{dk}{dr}\frac{r_k}{2} + \frac{r_k}{r^2}\left(r^2\frac{dk}{dr} + 2kr\right)\right]$$

$$= u_{rms}^3\left[\frac{r_k}{2}\frac{dk}{dr} + 2k\frac{r_k}{r^2}\right]$$

Now, taking a derivative with respect to r_k

$$\frac{\partial S_{ik,i}}{\partial r_k}(\underline{r},t) = u_{rms}^3 \frac{\partial}{\partial r_k}\left[2k\frac{r_k}{r^2} + \frac{r_k}{2}\frac{dk}{dr}\right]$$

$$= u_{rms}^3\left[\left(\frac{6}{r}k + 2r_k\frac{r_k}{r}\left(-\frac{1}{r^2}\right) + 2\frac{r_k}{r}\frac{r_k}{r}\frac{\partial k}{\partial r}\right) + \left(\frac{3}{2}\frac{dk}{dr} + \frac{1}{2}r_k\frac{r_k}{r}\frac{\partial^2 k}{\partial r^2}\right)\right]$$

$$= u_{rms}^3\left[\left(\frac{4}{r}k + 2\frac{\partial k}{\partial r}\right) + \left(\frac{3}{2}\frac{dk}{dr} + \frac{1}{2}r\frac{\partial^2 k}{\partial r^2}\right)\right]$$

$$= \frac{1}{2}u_{rms}^3\left[r\frac{\partial^2 k}{\partial r^2} + 7\frac{\partial k}{\partial r} + \frac{8}{r}k\right]$$

$$= \frac{1}{2r^2}u_{rms}^3\left[r^3\frac{\partial^2 k}{\partial r^2} + 7r^2\frac{\partial k}{\partial r} + 8rk\right]$$

$$= \frac{1}{2r^2}u_{rms}^3\left[\frac{\partial}{\partial r}\left(r^3\frac{\partial k}{\partial r}\right) - 3r^2\frac{\partial k}{\partial r} + 7r^2\frac{\partial k}{\partial r} + 8rk\right]$$

$$\frac{\partial S_{ik,i}}{\partial r_k}(\underline{r},t) = \frac{1}{2r^2}u_{rms}^3\left[\frac{\partial}{\partial r}\left(r^3\frac{\partial k}{\partial r}\right) + 4\left(\frac{\partial}{\partial r}(r^2k) - 2rk\right)r^2\frac{\partial k}{\partial r} + 8rk\right]$$

$S_{ik,i}(\underline{r},t) = -S_{ik,i}(-\underline{r},t)$ since $k(r) = -k(-r)$

$$\frac{\partial S_{ik,i}}{\partial r_k}(\pm\underline{r},t) = \pm\frac{1}{2r^2}u_{rms}^3\frac{\partial}{\partial r}\left[r^3\frac{\partial k}{\partial r} + 4r^2k\right]$$

Term 4:

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r},t) = \overline{u^2}\left[7\frac{\partial^2 f}{\partial r^2} + \frac{8}{r}\frac{\partial f}{\partial r} + r\frac{\partial^3 f}{\partial r^3}\right]$$

Chapter 4 Part 3 Eq. (7)

$\times \frac{r^2}{r^2}$

$$= \frac{\overline{u^2}}{r^2}\left[7r^2\frac{\partial^2 f}{\partial r^2} + 8r\frac{\partial f}{\partial r} + r^3\frac{\partial^3 f}{\partial r^3}\right]$$

$$= \frac{\overline{u^2}}{r^2}\left[7\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - 14r\frac{\partial f}{\partial r} + 8r\frac{\partial f}{\partial r} + \frac{\partial}{\partial r}\left(r^3\frac{\partial^2 f}{\partial r^2}\right) - 3r^2\frac{\partial^2 f}{\partial r^2}\right]$$

$$= \frac{\overline{u^2}}{r^2}\left[7\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \cancel{6r\frac{\partial f}{\partial r}} + \frac{\partial}{\partial r}\left(r^3\frac{\partial^2 f}{\partial r^2}\right) - 3\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \cancel{6r\frac{\partial f}{\partial r}}\right]$$

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r},t) = \frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]$$

Therefore, Eq. (1) becomes,

$$\underbrace{\frac{\partial}{\partial t}\left[\overline{u^2}\frac{1}{r^2}\frac{d(r^3 f)}{dr}\right]}_{\boxed{1}} = \underbrace{2\frac{1}{2r^2}u_{rms}^3\frac{\partial}{\partial r}\left[r^3\frac{\partial k}{\partial r} + 4r^2 k\right]}_{\boxed{2+3}} + \underbrace{2\nu\frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]}_{\boxed{4}}$$

$$\cancel{\frac{1}{r^2}}\frac{\partial}{\partial t}\left[\overline{u^2}\frac{d(r^3 f)}{dr}\right] = \cancel{\frac{1}{r^2}}u_{rms}^3\frac{\partial}{\partial r}\left[r^3\frac{\partial k}{\partial r} + 4r^2 k\right] + 2\nu\cancel{\frac{1}{r^2}}\overline{u^2}\frac{\partial}{\partial r}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]$$

Integrate over r

$$\frac{\partial}{\partial t}\left[\overline{u^2}r^3 f\right] = u_{rms}^3\left[r^3\frac{\partial k}{\partial r} + 4r^2 k\right] + 2\nu\overline{u^2}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]$$

Divide by r^3

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = u_{rms}^3\left[\frac{\partial k}{\partial r} + \frac{4}{r}k\right] + 2\nu\overline{u^2}\left[\frac{4}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\right] \quad (2)$$

Karman-Howarth Equation

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (3)$$

Alternative version Pope 6.75 Appendix A.1

The Karman-Howarth equation relates $f(r,t), k(r,t)$ and $u_{rms}(t)$. However, as with the $\mathcal{R}_{ij}$ equation the K-H equation is not closed, as if considered an equation for $f(r,t)$, it contains an additional unknown, i.e., the triple velocity correlation term $k(r,t)$.

Using a Taylor expansion for f and k

$$f(r,t) = \underbrace{f(0,t)}_{\boxed{1}} + f''(0,t)\frac{r^2}{2!} + f^{IV}(0,t)\frac{r^4}{4!} + \ \ldots$$

$$k(r,t) = \underbrace{k'(0,t)}_{\boxed{=0}}\, r + k'''(0,t)\frac{r^3}{3!} + \cdots$$

and substituting into Eq. (3) gives two equations by gathering terms depending on like powers of r. Appendix A.2

The r^0 equation gives,

$$\frac{d\overline{u^2}}{dt} = 10\nu\overline{u^2}f''(0,t) \quad (4)$$

And using the definitions of turbulent kinetic energy

$$k = \frac{3}{2}\overline{u^2}$$

and Taylor microscale

$$\lambda_f^2 = -\frac{2}{f''(0,t)}$$

in Eq. (4) yields

$$\frac{dk}{dt} = \frac{3}{2} 10\nu\overline{u^2}\left(-\frac{2}{\lambda_f^2}\right) = -30\frac{\nu\overline{u^2}}{\lambda_f^2}$$

And using the TKE equation for homogeneous isotropic turbulence

$$\frac{dk}{dt} = -\varepsilon$$

Gives

$$\varepsilon = 30\frac{\nu\overline{u^2}}{\lambda_f^2} = 15\frac{\nu\overline{u^2}}{\lambda_g^2}$$

A result obtained already in Chapter 4.

The r^2 equation gives,

$$\frac{d\varepsilon}{dt} = S_k^* R_T^{1/2} \frac{\varepsilon^2}{k} - G^* \frac{\varepsilon^2}{k}$$

Which, coupled with

$$\frac{dk}{dt} = -\varepsilon$$

Describes the decay of homogeneous isotropic turbulence, as per Part 2. This shows that all the isotropy information in the k and ε equations is contained in the Karman-Howart equation, for which it should be emphasized were derived from the Navier-Stokes equations, as were the k and ε equations.

Assuming self-similarity

$$f(r,t) = \tilde{f}\left(\frac{r}{L(t)} = \eta\right)$$

$$k(r,t) = \tilde{k}\left(\frac{r}{L(t)} = \eta\right)$$

$L(t) = \lambda_g(t)$

The Karman-Howarth equation becomes,

$$2\eta^{-4}\frac{d}{d\eta}\left(\eta^4 \frac{d\tilde{f}}{d\eta}\right) + \eta\frac{d\tilde{f}}{d\eta}\left(\frac{7}{3}G_0 - 5\right) + 10\tilde{f} = R_\lambda\left(\frac{7}{6}S_{k_0}\eta\frac{d\tilde{f}}{d\eta} - \eta^{-4}\frac{d\left(\eta^4\tilde{k}\right)}{d\eta}\right) \quad (5)$$

Appendix A.3

Where $\eta = r/\lambda_g$ is a similarity variable.

Eq. (5) represents a single ODE for $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ with $R_\lambda(t)$ acting as a parameter.

Note that Eq. (5) contains G_0 and S_{k_0}, which were shown in Part 2 to be constants during self-similar decay and equal to:

$$G = \tilde{f}^{IV}(0)$$

$$-S_k = \tilde{k}'''(0)$$

For self-similarity $f(r,t) = \tilde{f}(\eta) \neq f(t)$ and $k(r,t) = \tilde{k}(\eta)) \neq f(t)$. Consequently, both Eq. (5) RHS and LHS = 0 for $R_\lambda(t) \neq 0$, otherwise LHS multivalued for $R_\lambda(t)$, since $\tilde{f}$, $\tilde{k}$, S_{k_0}, $G_0 \neq f(t)$.

However, if $R_\lambda(t)$ is sufficiently small, then RHS ≈ 0 such that LHS = 0, which is referred to as the separability condition.

LHS = 0 gives the confluent hypergeometric equation with solution

$$\tilde{f}(\eta) = M\left(\frac{1}{G_0^* - 1}, \frac{5}{2}, -\frac{5(G_0^* - 1)}{4}\eta^2\right) \quad (6)$$

Where M is the confluent hypergeometric function.

Integration of the RHS=0 of Eq. (5) yields

$$\tilde{k}(\eta) = \frac{7}{6} S_{k_0} \frac{1}{\eta^4} \int_0^\eta s^5 \frac{d\tilde{f}}{ds} ds \quad (7)$$

Which can be solved using $\tilde{f}$ given by Eq. (6).

Self-similarity provides $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ for chosen G_0 and S_{k_0}. Recall complete similarity not possible in isotropic decay ∴ in reality $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ must be $f(t)$. However, still useful to examine high and low R_T equilibrium solutions.

1) For small R_λ, which is still $f(t)$ near $R_{T_\infty} = 0 \rightarrow R_{T_0} \sim 0.1$. RHS of Eq. (5) small and using $G_0^* = 7/5$ for final period

$$\tilde{f}(\eta) = M\left(\frac{5}{2}, \frac{5}{2}, -\frac{\eta^2}{2}\right) = e^{-\frac{\eta^2}{2}} \quad (8)$$

In agreement with Part 2

$$f(r,t) = e^{-\frac{r^2}{2\lambda_g^2}}$$

Assuming that decay is self-similar near $R_T = 0$, then Eq. (7) holds and solving the integral yields,

$$\tilde{k}(\eta) = \frac{7}{6} S_{k_0} \frac{1}{\eta^4}\left[(\eta^5 + 5\eta^3 + 15\eta)e^{-\eta^2/2} - 15\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right] \quad (9)$$

where

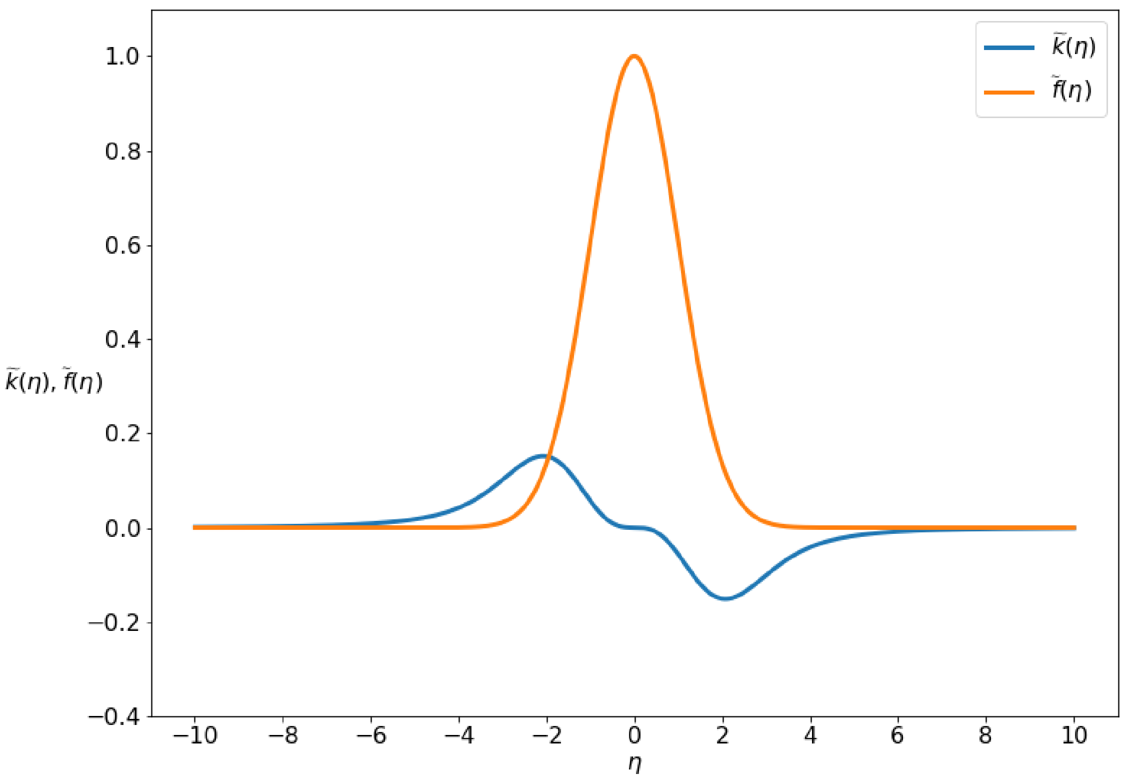
$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-s^2} ds$$

is the error function.

Note that G_0^* and S_{k_0} are constants based on assigned values provided by EFD or DNS.

Fig. 5.7 shows a plot of Eq. (9), where $\tilde{k}$ is seen to have a much slower decay for large η than the Gaussian form of $\tilde{f}$.

$$\tilde{k}(\eta) \to \eta^{-4} \quad \text{for} \quad \eta \to \infty$$



2) For large Re equilibrium, consider $R_\lambda =$ constant $\neq 0$. If $R_\lambda = f(t)$, LHS would have to be multivalued to satisfy equation. If $R_\lambda =$ constant a solution for and $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ exists, but the equation in indeterminate (1eq. 2 unknowns; thus, unlike self-similar k and ε equation, Karman-Howarth equation is not solvable without additional assumptions.

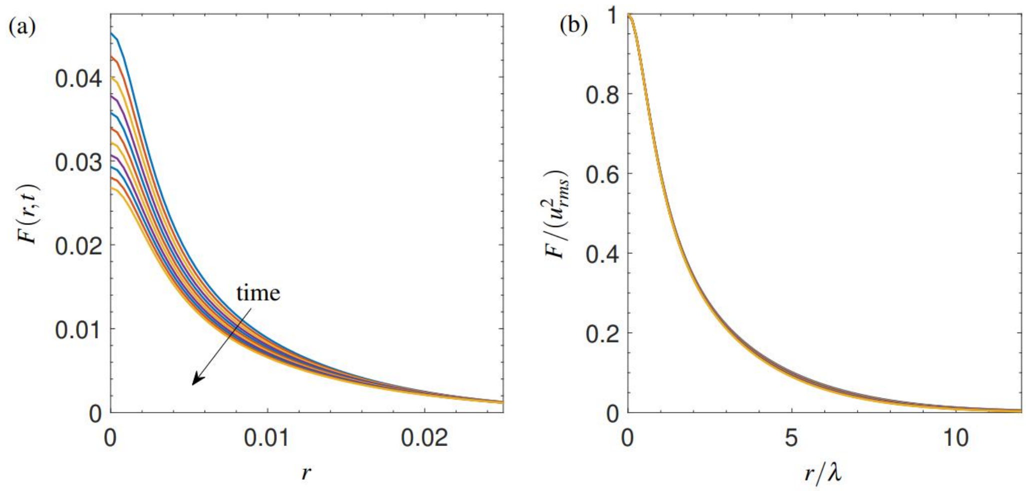


Figure 3. Unscaled (a) and scaled (b) double correlations. Increasing time corresponds to decreasing magnitude in (a), while profiles collapse in (b) under the scaling. All data is Case 2, while Cases 1 and 3 show similar behavior.

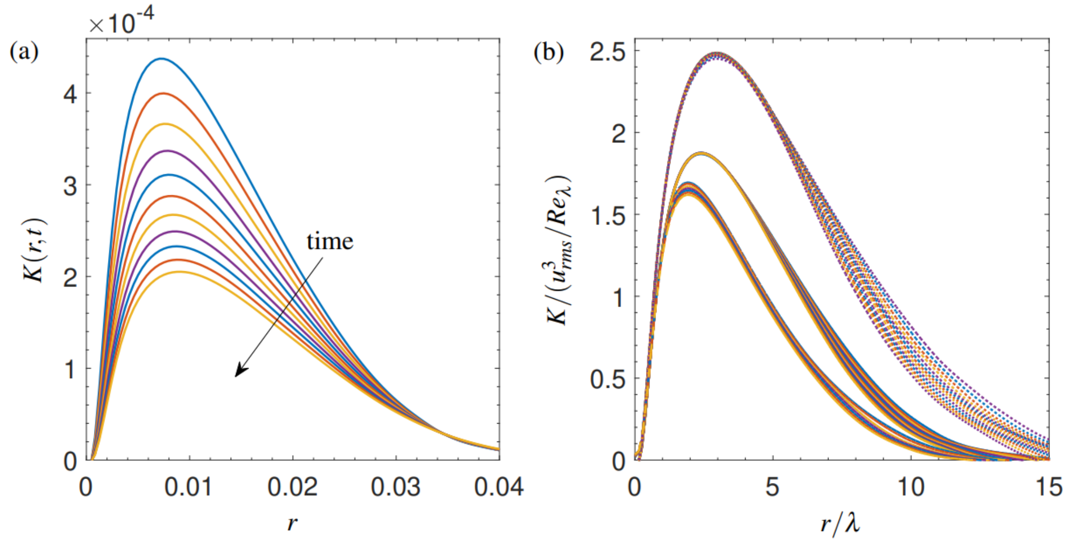


Figure 4. Unscaled (a) and scaled (b) triple correlations. Increasing time corresponds to decreasing magnitude in (a), while profiles collapse in (b) under the scaling, where Case 1 is shown in dashed lines (lowest magnitude), Case 2 in solid, and Case 3 in dotted lines (highest magnitude). The scaling works by accounting for the decreasing Reynolds number in each simulation but not the differences in Reynolds number between cases, which remains a point of ongoing investigation.

Byers, C. P., MacArt, J. F., Mueller, M. E., & Hultmark, M. (2019). Similarity constraints in decaying isotropic turbulence. Paper presented at 11th International Symposium on Turbulence and Shear Flow Phenomena, TSFP 2019, Southampton, United Kingdom.

Additional Discussion Karman-Howarth Equation (Pope pp. 202-205)

$$\frac{\partial}{\partial t}\left[\overline{u^2} f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (3)$$

a) Closure problem, i.e., one equation, two unknowns $\tilde{f}(\eta)$ and $\tilde{k}(\eta) \rightarrow$ one could write equation for $\tilde{k}(\eta)$, but it would depend on fourth-order correlation and so on.

b) Terms in k and ν represent inertial and viscous processes, respectively.

c) At $r = 0$, k term $= 0$ since

$$k(r,t) \approx k''' r^3/3! + k^V r^5/5!$$

And continuity implies $k'(0,t) = 0$; also, f is even in r, Eq. (3) becomes,

$$\frac{\partial}{\partial t}\left[\overline{u^2} f\right]\Big|_{r=0} = 2\nu\overline{u^2}\left[\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)\right]_{r=0}$$

$$= 2\nu\overline{u^2}\frac{1}{r^4}\left[4r^3\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}\right]_{r=0}$$

$$= 2\nu\overline{u^2}\frac{1}{r^4}\left[4r^4\frac{1}{r}\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}\right]_{r=0} = 2\nu\overline{u^2}\frac{1}{r^4}\left[5r^4\frac{\partial^2 f}{\partial r^2}\right]_{r=0}$$

$f(0,t) = 1$

$$\frac{d}{dt}\overline{u^2} = 10\nu\overline{u^2} f''(0,t) = -\frac{10\nu\overline{u^2}}{\lambda_g(t)^2} = -\frac{2}{3}\varepsilon \quad (10)$$

$$\lambda_g(t)^2 = -\frac{1}{f''(0,t)}$$

Pope Ex. 6.6

Where the Taylor expansion for $f(r)$

$$f'(r) = \cancel{f'(0)} + rf''(0) + \frac{r^2}{\cancel{2!}}\cancel{f'''}(0) + \cdots$$

$$\lim_{r \to 0} \frac{f'(r)}{r} = f''(0)$$

was used.

Hence, for $r = 0$, the Karman-Howart equation reduces to $\frac{2}{3}$ times the k equation,

$$\frac{dk}{dt} = -\varepsilon$$

d) Energy cascade for high Re hypothesis is that the energy transfer from larger to smaller scales is an inertial process for $r \gg \eta$, consequently, k term is responsible for this process.

e) If $\underline{u}(\underline{x}, t)$ were a Gaussian field then $k(r, t)$, like all higher order moments, would be zero → energy cascade depends on non-Gaussian aspects of the velocity field. This fact is used in the Quasi-normal approximation method for KH equation.

Skewness of velocity derivative

$$\overline{u^3}k'''(0,t) = \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3} = -S_k\left(\frac{\varepsilon}{15\nu}\right)^{3/2} = -\frac{2}{35}\overline{\omega_i\omega_j\frac{\partial u_i}{\partial x_j}}$$

where

$$S_k = -\frac{\overline{\left(u_{1,1}\right)^3}}{\overline{\left(u_{1,1}\right)^2}^{3/2}}$$

In the velocity-derivative skewness, includes – sign as per Bernard. Pope and Hinze define S_k without the − sign. In Bernard definition, S_k is positive, while for Hinze and Pope it is negative. This fact does not change the physical meaning of the equations but could require some sign changes in the derivations. Throughout these notes, Bernard definition is used to be consistent.

∴ connection between S_k, vortex stretching and transfer of energy between different scales, as will be shown in Part 4.

The Kolmogorov 4/5 law

The Karman-Howarth equation can be re-expressed in terms of the structure functions $D_{LL}(r,t)$ and $D_{LLL}(r,t)$

$$D_{LLL}(r,t) = \overline{\left[u_1\left(\underline{x} + r\hat{e}_1, t\right) - u_1\left(\underline{x}, t\right)\right]^3}$$

As

$$\frac{\partial}{\partial t} D_{LLL} + \frac{1}{3r^4}\frac{\partial}{\partial r}\left(r^4 D_{LLL}\right) = \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial D_{LLL}}{\partial r}\right) - \frac{4}{5}\varepsilon$$

Pope Ex. 6.9

Integrating

$$\frac{3}{r^5}\int_0^r s^4 \frac{\partial}{\partial t} D_{LLL}(s,t)ds = 6\nu \frac{\partial D_{LLL}}{\partial r} - D_{LLL} - \frac{4}{5}\varepsilon r$$

For isotropic turbulence in the inertial subrange, unsteady term = 0 and viscous term negligible leads to Kolmogorov -4/5 law

$$D_{LLL} = -\frac{4}{5}\varepsilon r$$

Kolmogorov also argued that the structure function skewness,

$$S' \equiv D_{LLL}(r,t)/D_{LL}(r,t)^{3/2}$$

is constant, leading to

$$D_{LL}(r,t) = \left(-\frac{4}{5S'}\right)^{2/3} (\varepsilon r)^{2/3}$$

Which represents Kolmogorov hypothesis, and shows consistency between it and the NS equations, and relates Kolmogorov constant to skewness S'.

The Loitsyanskii integral

Multiplying Eq. (3) (K-H equation Pope form) by r^4 and integrating between 0 and R, yields

$$\frac{d}{dt}\int_0^R \overline{u^2} r^4 f(r,t)dr = u_{rms}^3 R^4 k(r,t) + 2\nu\overline{u^2} r^4 f'(r,t) \quad (11)$$

Loitsyanskii considered $\lim_{R\to\infty}$ Eq. (11), assuming that $f(r,t)$ and $k(r,t)$ decrease rapidly with r, such that the Loitsyanskii integral,

$$B_2 = \int_0^\infty \overline{u^2} r^4 f(r,t)dr$$

Converges, such that terms in $k(R,t)$ and $f'(R,t)$ vanish.

With these assumptions, $B_2 \neq f(t)$, and became known as Loitsyanskii invariant.

However, these assumptions are incorrect, as shown by Saffman, who considered the following invariant,

$$C = \int_0^\infty r^2 \mathcal{R}(r)dr$$

where

$$\mathcal{R}(r) = \frac{1}{8\pi r^2}\int_{|\underline{r}|=r} \mathcal{R}_{ii}(\underline{r})dA(\underline{r})$$

And for isotropic turbulence, $\mathcal{R}(r) = \frac{1}{2}\overline{u^2}(3f + rf')$.

Kolmogorov, starting from the invariance of the Loitsyanskii integral, obtained that for isotropic turbulence $\overline{u^2} \propto t^{-10/7}$ (1.43) during decay.

During the final decay predictions of the $-5/2$ (-2.5) law are in agreement. However, for higher Re conditions, approaching or including inertial subrange, there is no consensus, and many solutions are obtained depending on the approach and assumptions used.

Bachelor, Decay of turbulence in the final period (1948), as shown by Bernard: t^{-1}

Saffman, imposing the invariance of C, obtained

$$\overline{u^2} = KC^{2/5}t^{-6/5} \text{ (1.2)}$$

Where K is a constant that depends upon the structure of the turbulence. To obtain this result, differently from Kolmogorov, Saffman only required self-similarity, not isotropy.

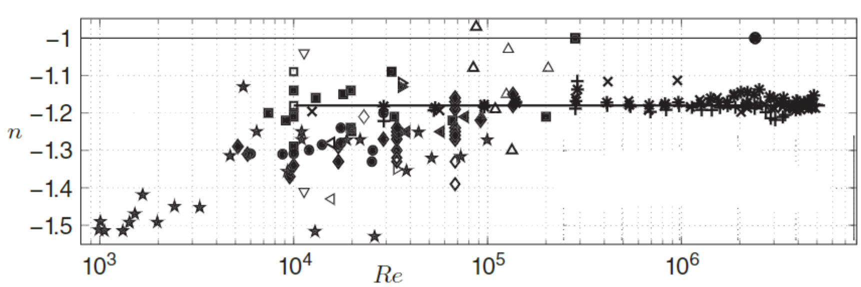


Figure 5.6 Measured power law exponents in decaying homogeneous turbulence from numerous experiments [9]. Filled symbols represent traditional turbulence behind a grid of bars. Open symbols are other turbulence sources. The symbols with ×,*, and + are from three different probes used in decaying turbulence behind a grid of bars for which the Reynolds number was changed only by altering the viscosity of the working fluid. Used by permission of AIP.

Hinze Section 3.3

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)$$

Karman-Howarth equation represents one equation in two unknowns $f(r)$ and $k(r)$. Similar to NS equations leads to closure problem since the number of unknowns is larger than the number of equations. Also, as with NS equations if one obtains higher order velocity correlation equation, they lead to additional unknowns.

Truncation approximation= neglect higher order terms but leads to unphysical solutions.

Quasi-normal approximation= neglect higher order cumulants = assume Gaussian 4th order correlation. However, this implies $S_{ijl} = 0$, which is unacceptable ∴ again leads to unphysical solutions.

Direct-interaction approximation= considers interaction of eddies of different sized, including their randomness.

Before these approaches, closure problem also attacked based on physical assumptions for the inertial term.

Consider Eq. (1) using the simplification $S_{ik,i}(r,t) = -S_{ik,i}(-r,t)$

$$\frac{\partial \mathcal{R}_{ii}}{\partial t}(\underline{r},t) - 2\frac{\partial S_{ik,i}}{\partial r_k}(-\underline{r},t) = 2\nu\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r},t) \quad (12)$$

In Hinze, the following notation is used.

$$\mathcal{R}_{ii} = Q_{i,i} \qquad 2\frac{\partial S_{ik,i}}{\partial r_k}(-r,t) = S_{i,i}$$

1. Taking the second moment of each term in Eq. (12)

$$\frac{\partial}{\partial t}\int_0^\infty dr\, r^2 \mathcal{R}_{ii} - 2\int_0^\infty dr\, r^2 \frac{\partial S_{ik,i}}{\partial r_k} = 2\nu \left(r^2 \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2} \right)\Bigg|_0^\infty$$

Applying incompressibility yields

$$\int_0^\infty dr\, r^2 \mathcal{R}_{ii} = 2\int_0^\infty dr\, r^2 \frac{\partial S_{ik,i}}{\partial r_k} = 0$$

Consequently

$$\lim_{r\to\infty} r^2 \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2} = 0 \qquad (13)$$

Which shows how fast $\mathcal{R}_{ii}$ decreases as r increases.

2. Multiplying Eq. (3) by r^4 and taking the fourth moment of each term

$$\frac{\partial}{\partial t}\left(\overline{u^2}\int_0^\infty dr\, r^4 f\right) = (u_{rms}^3 r^4 k)|_0^\infty + 2\nu\overline{u^2}\left(r^4\frac{\partial f}{\partial r}\right)\Bigg|_0^\infty$$

With certain assumptions concerning large-scale structure of turbulence, it is reasonable to expect that,

$$\lim_{r\to\infty}\left(r^4\frac{\partial f}{\partial r}\right) = 0 \quad (14)$$

See Hinze pp. 207, 216-218

$\mathcal{R}_{ii}$ and f have same asymptotic behavior, so if $f(r)$ behaves like r^{-n} for large r, Eq. (13) requires $n > 1$. Eq. (14), instead, would require $n > 3$.

The term $r^4 k$ has usually been assumed to approach zero for increasing r, and this assumption has been used by Loitsyanskii to obtain,

$$B_2 = \int_0^\infty \overline{u^2} r^4 f(r,t) dr$$

must be invariant and not function of time→ Loitsyanskii invariant. However, this is not true, and depends on IC of the turbulence (see Saffman).

Consider now limiting case of Eq. (12), where the viscosity effects become predominant → characteristic of final decay,

$$2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r}, t) \gg 2 \frac{\partial S_{ik,i}}{\partial r_k}(-\underline{r}, t)$$

This hypothesis allows to treat the vector $\underline{r}$ as a scalar r, such that $\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2} = \frac{\partial^2 \mathcal{R}_{ii}}{\partial r}$

$$\frac{\partial \mathcal{R}_{ii}}{\partial t}(r, t) = 2\nu \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial \mathcal{R}_{ii}}{\partial r}(r, t)\right] \quad (15)$$

Where the identity

$$2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(r, t) = 2\nu \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial \mathcal{R}_{ii}}{\partial r}(r, t)\right]$$

was used.

Assume

$$\mathcal{R}_{ii}(r, t) = \varphi(t)\psi(\chi)$$

Where $\chi = r/\sqrt{8\nu t}$, i.e., separation of variables

Substituting into Eq. (15) leads to two differential equations,

$$\frac{1}{\varphi}\frac{d\varphi}{dt} = -\frac{\alpha}{t}$$

With solution $\varphi = c \times t^{-\alpha}$ and

$$\chi\frac{d^2\psi}{d\chi^2} + 2(\chi^2 + 1)\frac{d\psi}{d\chi} + 4\alpha\chi\psi = 0$$

Assuming $\alpha = (2p + 1)/2$ with p integer, the solution is

$$\psi_p = \frac{1}{\chi}\exp(-\chi^2)H_{2p-1}(\chi)$$

Where $H_n(\chi)$ is the Hermite polynomial

$$H_n(\chi) = (-1)^n \exp(\chi^2)\frac{d^n}{d\chi^n}\exp(-\chi^2)$$

$$H_0 = 1, \qquad H_1 = 2\chi, \qquad H_2 = 4\chi^2 - 2, \qquad H_3 = 8\chi^3 - 12\chi$$

General solution of Eq. (15) is:

$$\mathcal{R}_{ii}(r,t) = \frac{\sqrt{8\nu}}{r}\exp(-r^2/8\nu t)\sum_{1}^{\infty}\frac{A_p}{t_p}H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right)$$

Where the constants A_p must be chosen such that the series converges and that

$$\mathcal{R}_{ii}(0,t) = \overline{u^2}\left(3\underbrace{f(0)}_{\boxed{1}} + r\cancel{\frac{df}{dr}(0)}\right) = 3\overline{u^2}$$

Applying 2nd moment condition

$$\int_0^\infty dr\, r^2 \mathcal{R}_{ii} = 0$$

Gives $A_1 = 0.$

Moreover, substituting $\mathcal{R}_{ii}(r,t) = \overline{u^2}(3f + rf')$ it can be seen that:

1. $A_1 = 0$
2. Using 4th moment condition (Loitsyanskii integral) $\Rightarrow$ only $p = 2$ term $\neq 0.$

Therefore, solution of Eq. (15) for $p = 2$ may be reduced to

$$\mathcal{R}_{ii}(r,t) = -\frac{4A_2}{t^{5/2}}\left(3 - \frac{r^2}{4\nu t}\right)\exp(-r^2/8\nu t)$$

And applying the condition $\mathcal{R}_{ii}(0,t) = 3\overline{u^2}$ yields

$$\overline{u^2} = -4A_2 t^{-\frac{5}{2}} = c \times t^{-\frac{5}{2}} \quad (16)$$

And, consequently,

$$\mathcal{R}_{ii}(r,t) = c \times t^{-5/2}\left(3 - \frac{r^2}{4\nu t}\right)\exp(-r^2/8\nu t)$$

$$\mathcal{R}_{ii}(r,t) = \overline{u^2}\left(3 - \frac{r^2}{4\nu t}\right)e^{-\frac{r^2}{8\nu t}} \quad (17)$$

Combining the relation between $\mathcal{R}_{ii}(r,t)$ and $f(r,t)$

$$\mathcal{R}_{ii}(r,t) = \overline{u^2}\left(3f(r,t) + r\frac{\partial f}{\partial r}(r,t)\right) = \frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}[r^3 f(r,t)]$$

With Eq. (17) gives the following differential equation

$$\frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}[r^3 f(r,t)] = \overline{u^2}\left(3 - \frac{r^2}{4\nu t}\right)e^{-\frac{r^2}{8\nu t}}$$

With boundary conditions

$$f(0,t) = 1$$

And by integration

$$r^3 f(r,t) = \int r^2\left(3 - \frac{r^2}{4\nu t}\right)e^{-\frac{r^2}{8\nu t}} = r^3 e^{-\frac{r^2}{8\nu t}} + C$$

Or equivalently

$$f(r,t) = e^{-\frac{r^2}{8\nu t}} \quad (18)$$

Where $C = 0$ from the application of BCs.

Thus, for dominating viscosity effects, Eq. (12) shows the decay law for the turbulence ⇒ $-5/2$ decay as shown in Chapter 5 Part 2.

$f(r,t)$ has shape Gaussian curve and remains self-preserving during decay. Shows good agreement with EFD. Moreover, using Eqs. (16) and (18) to evaluate Loitsyanskii integral proves that it is an exact invariant with respect to time, in these conditions. Appendix A.4

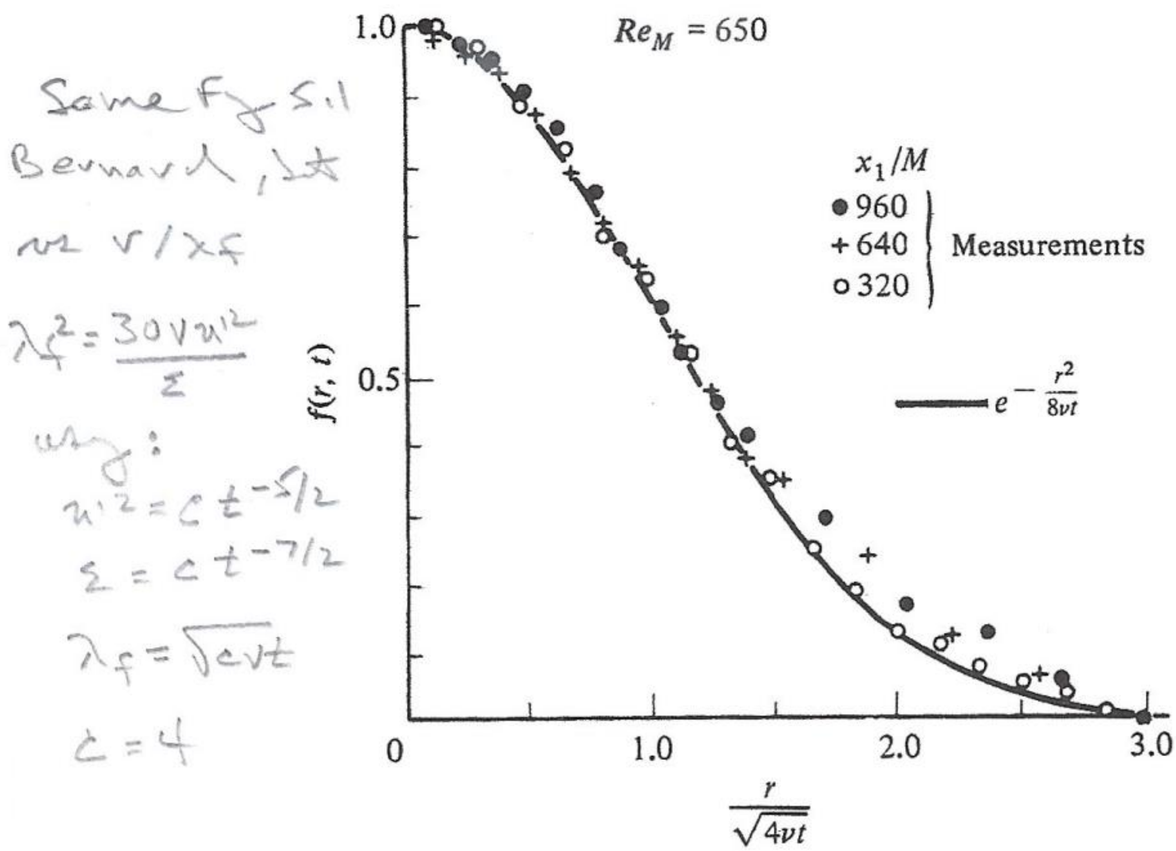


FIGURE 3-11
Longitudinal-velocity-correlation coefficient f(r,t) in the final period of decay. (From: *Batchelor, G. K., and A. A. Townsend,*[7] *by permission of the Royal Society.*)

Appendix A

Note: To avoid confusion between $k(r,t)$ and TKE, a capital K will be used for the TKE, in this Appendix.

A.1

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = u_{rms}^3\left[\frac{\partial k}{\partial r} + \frac{4}{r}k\right] + 2\nu\overline{u^2}\left[\frac{4}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\right]$$

Multiply and divide by r^4

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\left[r^4\frac{\partial k}{\partial r} + 4r^3k\right] + 2\nu\overline{u^2}\left[4r^3\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}\right]$$

$$= \frac{u_{rms}^3}{r^4}\left[\frac{\partial(r^4k)}{\partial r} - \cancel{4r^3k} + \cancel{4r^3k}\right] + \frac{2\nu\overline{u^2}}{r^4}\left[\underbrace{4r^3\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}}_{\boxed{\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)}}\right]$$

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)$$

A.2

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (1A)$$

Taylor series for $f(r,t)$ and $k(r,t)$

$$f(r,t) = \underbrace{f(0,t)}_{\boxed{1}} + f''(0,t)\frac{r^2}{2!} + f^{IV}(0,t)\frac{r^4}{4!} + \ \dots$$

$$k(r,t) = \underbrace{k'(0,t)}_{\boxed{=0}}r + k'''(0,t)\frac{r^3}{3!} + \cdots$$

Substitute into Eq. (1A)

$$\frac{\partial}{\partial t}\left[\overline{u^2}\left(1 + f''(0,t)\frac{r^2}{2!} + f^{IV}(0,t)\frac{r^4}{4!}\right)\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}\left(r^4 k'''(0,t)\frac{r^3}{3!}\right) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\left(f''(0,t)r + f^{IV}(0,t)\frac{r^3}{3!}\right)\right)$$

$$\frac{\partial\overline{u^2}}{\partial t} + \frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2}f''(0,t)\right) + \frac{r^4}{4!}\frac{\partial}{\partial t}\left(f^{IV}(0,t)\right) = \frac{u_{rms}^3}{r^4}\left(\frac{7}{6}r^6 k'''(0,t)\right) + \frac{2\nu\overline{u^2}}{r^4}\left(5r^4 f''(0,t) + \frac{7}{6}r^6 f^{IV}(0,t)\right)$$

Simplify powers or r

$$\frac{\partial \overline{u^2}}{\partial t} + \frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2} f''(0,t)\right) + \frac{r^4}{4!}\frac{\partial}{\partial t}\left(f^{IV}(0,t)\right) = \frac{7}{6}u_{rms}^3 r^2 k'''(0,t) + 10\nu\overline{u^2} f''(0,t) + \frac{7}{3}\nu\overline{u^2} r^2 f^{IV}(0,t)$$

Now, gather terms according to power of r

$$\boldsymbol{r^0}: \frac{\partial \overline{u^2}}{\partial t} = 10\nu\overline{u^2} f''(0,t) \quad (2A)$$

$$\boldsymbol{r^2}: \frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2} f''(0,t)\right) = \frac{7}{6}u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu\overline{u^2} r^2 f^{IV}(0,t) \quad (3A)$$

Only one term on the LHS for r^4, no need to consider it for this analysis.

Focus on Eq. (2A)

$$\frac{\partial \overline{u^2}}{\partial t} = 10\nu\overline{u^2} f''(0,t)$$

Using the definitions of turbulent kinetic energy

$$K = \frac{3}{2}\overline{u^2}$$

And Taylor microscale

$$\lambda_f^2 = -\frac{2}{f''(0,t)}$$

In Eq. (2A) yields

$$\frac{dK}{dt} = \frac{3}{2}10\nu\overline{u^2}\left(-\frac{2}{\lambda_f^2}\right) = -30\frac{\nu\overline{u^2}}{\lambda_f^2}$$

And using the TKE equation for homogeneous isotropic turbulence

$$\frac{dK}{dt} = -\varepsilon$$

Gives

$$\varepsilon = 30\frac{\nu\overline{u^2}}{\lambda_f^2} = 15\frac{\nu\overline{u^2}}{\lambda_g^2}$$

Focus on Eq. (3A)

$$\frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2}f''(0,t)\right) = \frac{7}{6}u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu\overline{u^2}r^2 f^{IV}(0,t)$$

$$\frac{r^2}{2}\left(\frac{\partial\overline{u^2}}{\partial t}f''(0,t) + \overline{u^2}\frac{\partial f''(0,t)}{\partial t}\right) = \frac{7}{6}u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu\overline{u^2}r^2 f^{IV}(0,t) \quad (4A)$$

Now focus on term in parenthesis on LHS

$$\frac{\partial \overline{u^2}}{\partial t} f''(0,t) + \overline{u^2}\frac{\partial f''(0,t)}{\partial t} = \frac{2}{3}\frac{\partial K}{\partial t} f''(0,t) + \frac{2}{3} K \frac{\partial f''(0,t)}{\partial t} \qquad (5A)$$

Substituting

$$\frac{dK}{dt} = -\varepsilon$$

Into Eq. (5A)

$$-\frac{2}{3}\varepsilon f''(0,t) + \frac{2}{3} K \frac{\partial f''(0,t)}{\partial t} \qquad (6A)$$

And using

$$f''(0,t) = -\frac{\varepsilon}{15\nu\overline{u^2}} = -\frac{\varepsilon}{10\nu K}$$

Into Eq. (6A) gives

$$\frac{2}{3}\frac{\varepsilon^2}{10\nu K} - \frac{2}{3} K \frac{d}{dt}\left(\frac{\varepsilon}{10\nu K}\right) = \frac{\varepsilon^2}{15\nu K} - \frac{K}{15\nu}\left(\frac{1}{K}\frac{d\varepsilon}{dt} - \frac{\varepsilon}{K^2}\underbrace{\frac{dK}{dt}}_{\boxed{-\varepsilon}}\right)$$

$$= \frac{\varepsilon^2}{15\nu K} - \frac{K}{15\nu}\left(\frac{1}{K}\frac{d\varepsilon}{dt} + \frac{\varepsilon^2}{K^2}\right) = \cancel{\frac{\varepsilon^2}{15\nu K}} - \frac{1}{15\nu}\frac{d\varepsilon}{dt} - \cancel{\frac{\varepsilon^2}{15\nu K}}$$

Therefore, it was proved that,

$$\frac{\partial \overline{u^2}}{\partial t} f''(0,t) + \overline{u^2}\frac{\partial f''(0,t)}{\partial t} = -\frac{1}{15\nu}\frac{d\varepsilon}{dt} \qquad (7A)$$

Substituting Eq. (7A) into Eq. (4A) gives

$$-\frac{r^2}{2}\left(\frac{1}{15\nu}\frac{d\varepsilon}{dt}\right) = \frac{7}{6}u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu\overline{u^2}r^2 f^{IV}(0,t)$$

And isolating $d\varepsilon/dt$

$$\frac{d\varepsilon}{dt} = -35\nu u_{rms}^3 k'''(0,t) - 70\nu^2\overline{u^2}f^{IV}(0,t)$$

Which is equivalent to

$$\frac{d\varepsilon}{dt} = S_k^* R_T^{1/2}\frac{\varepsilon^2}{k} - G^*\frac{\varepsilon^2}{k}$$

As shown in Chapter 5 Part 1.

A.3

$$\frac{\partial}{\partial t}\left[\overline{u^2} f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (8A)$$

Self-similarity

$$f(r,t) = \tilde{f}(r/\lambda(t)) = \tilde{f}(r/L(t)) = \tilde{f}(\eta)$$

$$k(r,t) = \tilde{k}(r/\lambda(t)) = \tilde{k}(\eta)$$

$$\eta = r/\lambda$$

$$\frac{\partial\eta}{\partial\lambda} = \frac{\partial}{\partial\lambda}\left(\frac{r}{\lambda}\right) = -\frac{r}{\lambda^2} = -\frac{\eta}{\lambda} \quad (9A)$$

Focus on LHS of Eq. (8A)

$$\frac{\partial}{\partial t}\left[\overline{u^2}\tilde{f}\right] = \frac{2}{3}\frac{dK}{dt}\tilde{f} + \frac{2}{3}K\frac{\partial\tilde{f}}{\partial t}$$

$$= \frac{2}{3}\frac{dK}{dt}\tilde{f} + \frac{2}{3}K\frac{\partial\tilde{f}}{\partial t} \quad (10A)$$

$$\frac{\partial\tilde{f}}{\partial t} = \frac{\partial\tilde{f}}{\partial\lambda}\underbrace{\frac{\partial\lambda}{\partial t}}_{\boxed{\dot{\lambda}}} = \frac{\partial\tilde{f}}{\partial\eta}\underbrace{\frac{\partial\eta}{\partial\lambda}}_{\boxed{-\frac{\eta}{\lambda}}}\dot{\lambda} = -\dot{\lambda}\frac{\eta}{\lambda}\frac{d\tilde{f}}{d\eta}$$

Therefore, Eq. (10A) becomes,

$$\frac{\partial}{\partial t}\left[\overline{u^2}\tilde{f}\right] = \frac{2}{3}\frac{dK}{dt}\tilde{f} - \frac{2}{3}K\dot{\lambda}\frac{\eta}{\lambda}\frac{d\tilde{f}}{d\eta}$$

Now, focus on RHS Eq. (8A)

$$\frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)$$

Assuming Self-similarity: $r = \eta\lambda_g$

$$\frac{u_{rms}^3}{(\eta\lambda_g)^4}\underbrace{\frac{\partial}{\partial r}}_{\boxed{\frac{1}{\lambda_g}\frac{d}{d\eta}}}\left((\eta\lambda_g)^4\tilde{k}\right) + \frac{2\nu\overline{u^2}}{(\eta\lambda_g)^4}\underbrace{\frac{\partial}{\partial r}}_{\boxed{\frac{1}{\lambda_g}\frac{d}{d\eta}}}\left((\eta\lambda_g)^4\frac{1}{\lambda_g}\frac{d\tilde{f}}{d\eta}\right)$$

$$\frac{u_{rms}^3}{\eta^4\lambda_g}\frac{d}{d\eta}(\eta^4\tilde{k}) + \frac{2\nu\overline{u^2}}{\eta^4\lambda_g^2}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

$$\sqrt{\frac{2}{3}}\frac{K^{3/2}}{\eta^4\lambda_g}\frac{d}{d\eta}(\eta^4\tilde{k}) + \frac{4\nu K}{3\eta^4\lambda_g^2}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

Therefore, Eq. (8A) becomes,

$$\frac{2}{3}\frac{dK}{dt}\tilde{f} - \frac{2}{3}K\dot{\lambda}_g\frac{\eta}{\lambda_g}\frac{d\tilde{f}}{d\eta} = \left(\frac{2}{3}\right)^{3/2}\frac{K^{3/2}}{\eta^4\lambda_g}\frac{d}{d\eta}(\eta^4\tilde{k}) + \frac{4\nu K}{3\eta^4\lambda_g^2}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

And multiplying by $3\lambda_g^2/2\nu K$

$$\underbrace{\frac{\lambda_g^2}{\nu K}\frac{dK}{dt}\tilde{f}}_{\boxed{1}} - \underbrace{\lambda_g\dot{\lambda}_g\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta}}_{\boxed{2}} = \underbrace{\left(\frac{2}{3}\right)^{\frac{1}{2}}\frac{\lambda_g K^{\frac{1}{2}}}{\nu\eta^4}\frac{d}{d\eta}(\eta^4\tilde{k})}_{\boxed{3}} + \frac{2}{\eta^4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right) \quad (11A)$$

Term 1:

$$\frac{\lambda_g^2}{\nu K}\frac{dK}{dt}\tilde{f} = -\frac{\lambda_g^2}{\nu K}\varepsilon\tilde{f} = -\frac{\lambda_g^2}{\nu K}\frac{10\nu K}{\lambda_g^2}\tilde{f} = -10\tilde{f}$$

Term 2:

$$-\lambda_g\dot{\lambda_g}\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = -\sqrt{\frac{10\nu K}{\varepsilon}}\frac{d}{dt}\left(\sqrt{\frac{10\nu K}{\varepsilon}}\right)\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = -\frac{1}{2}\frac{d}{dt}\left(\frac{10\nu K}{\varepsilon}\right)\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta}$$

$$= -5\eta\left(\underbrace{\frac{dK}{dt}}_{\boxed{-\varepsilon}}\frac{1}{\varepsilon} - \frac{K}{\varepsilon^2}\frac{d\varepsilon}{dt}\right)\frac{d\tilde{f}}{d\eta} = -5\eta\left(-1 - \frac{K}{\varepsilon^2}\frac{d\varepsilon}{dt}\right)\frac{d\tilde{f}}{d\eta}$$

Now substitute decay equation for ε

$$-\lambda_g\dot{\lambda_g}\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = -5\eta\left(-1 - \frac{K}{\varepsilon^2}\left(\frac{7}{3\sqrt{15}}S_{k_0}R_T^{\frac{1}{2}}\frac{\varepsilon^2}{k} - \frac{7}{15}G_0\frac{\varepsilon^2}{k}\right)\right)\frac{d\tilde{f}}{d\eta}$$

$$= 5\eta\frac{d\tilde{f}}{d\eta} + \frac{35}{3\sqrt{15}}\eta S_{k_0}R_T^{\frac{1}{2}}\frac{d\tilde{f}}{d\eta} - \frac{7}{3}\eta G_0\frac{d\tilde{f}}{d\eta} \quad (12A)$$

Recall relation between R_T and R_λ

$$\sqrt{R_T} = \sqrt{\frac{3}{20}}R_\lambda$$

And substitute into Eq. (12A)

$$-\lambda_g\dot{\lambda_g}\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = 5\eta\frac{d\tilde{f}}{d\eta} + \frac{7\cdot\cancel{5}}{3\sqrt{\cancel{5}\cdot\cancel{3}}}\eta S_{k_0}\sqrt{\frac{\cancel{3}}{\cancel{5}\cdot 2^2}}R_\lambda\frac{d\tilde{f}}{d\eta} - \frac{7}{3}\eta G_0\frac{d\tilde{f}}{d\eta}$$

$$= 5\eta\frac{d\tilde{f}}{d\eta} + \frac{7}{6}\eta S_{k_0}R_\lambda\frac{d\tilde{f}}{d\eta} - \frac{7}{3}\eta G_0\frac{d\tilde{f}}{d\eta}$$

Term 3:

$$\left(\frac{2}{3}\right)^{\frac{1}{2}}\frac{\lambda_g K^{\frac{1}{2}}}{\nu\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)=\frac{\lambda_g\sqrt{\overline{u^2}}}{\nu\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)=\frac{R_\lambda}{\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)$$

Therefore, Eq. (11A) becomes,

$$-10\tilde{f}+\frac{7}{6}\eta S_{k_0}R_\lambda\frac{d\tilde{f}}{d\eta}+\left(5-\frac{7}{3}G_0\right)\eta\frac{d\tilde{f}}{d\eta}=\frac{R_\lambda}{\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)+\frac{2}{\eta^4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

Reordering the term yields

$$\frac{2}{\eta^4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)+\eta\frac{d\tilde{f}}{d\eta}\left(\frac{7}{3}G_0-5\right)+10\tilde{f}=R_\lambda\left(\frac{7}{6}\eta S_{k_0}\frac{d\tilde{f}}{d\eta}-\eta^{-4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)\right)$$

A.4

Loitsyanskii integral

$$B_2 = \int_0^\infty \overline{u^2} r^4 f(r,t) dr$$

Using

$$\overline{u^2} = -4A_2 t^{-\frac{5}{2}} = c \times t^{-\frac{5}{2}}$$

$$f(r,t) = e^{-\frac{r^2}{8\nu t}}$$

And substituting into the expression for B_2

$$B_2 = c \times t^{-\frac{5}{2}} \int_0^\infty r^4 e^{-\frac{r^2}{8\nu t}} dr \qquad (13A)$$

Assuming full self-similarity, since the solution is in the final decay region, the variable η can be used to describe $f(r,t)$ such that,

$$f(r,t) = \tilde{f}\left(\frac{r}{\lambda(t)} = \eta\right)$$

Where $\lambda(t)$ represents the Taylor microscale, that varies with time:

$$\lambda(t) \propto \sqrt{t}$$

as shown in Chapter 5 Part 2.

Substituting $r = \eta\lambda$, $dr = \lambda d\eta$ in Eq. (13A) yields

$$B_2 = c \times t^{-\frac{5}{2}} \int_0^{\infty} \eta^4 \lambda^4 e^{-\frac{\eta^2\lambda^2}{8\nu t}} \lambda d\eta$$

$$B_2 = c \times \lambda^5 t^{-\frac{5}{2}} \int_0^{\infty} \eta^4 e^{-\frac{\eta^2\lambda^2}{8\nu t}} d\eta \qquad (14A)$$

Evaluating the integral in Eq. (14A) gives

$$\int_0^{\infty} \eta^4 e^{-a\eta^2} d\eta = \left[\frac{3\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{a}\eta\right)}{8a^{\frac{5}{2}}} - \frac{(2a\eta^3 + 3\eta)e^{-a\eta^2}}{4a^2}\right]_0^{\infty} \qquad (15A)$$

Where

$$a = \frac{\lambda^2}{8\nu t}$$

Evaluating Eq. (15A) at 0 and ∞ gives

$$\int_0^{\infty} \eta^4 e^{-a\eta^2} d\eta = \frac{3\sqrt{\pi}}{8a^{\frac{5}{2}}}$$

And substituting back into Eq. (13A)

$$B_2 = c \times \lambda^5 t^{-\frac{5}{2}} \frac{3\sqrt{\pi}}{8\left(\frac{\lambda^2}{8\nu t}\right)^{\frac{5}{2}}} = c \times \frac{\lambda^5 t^{-\frac{5}{2}}}{\lambda^5 t^{-\frac{5}{2}}} = c \neq f(t)$$