Chapter 4: Turbulence at Small Scales

Part 4: Inertial Subrange

Recall:

$$K(t) = \int_0^\infty E(k,t)dk$$

$E(k,t)$ shows how the TKE is distributed among the different scales of the flow.

k^{-1} =length scale of eddy associated with wave number k.

Richardson cascade, Kolmogorov hypotheses, theory of isotropic turbulence and dimensional analysis lead to the “most famous and prominent” feature of high Re turbulence: the universal power law form of the energy spectrum in the inertial subrange.

1. **Kolmogorov’s first similarity hypothesis.**
   In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions ($l < l_e$) have a universal form that is uniquely determined by ν and ε.

   In the universal equilibrium range, the turbulence is isotropic:
   $l < l_e, i.e., k > k_e = \frac{2\pi}{l_e}$

$$E = E(k, \varepsilon, \nu) \quad [m^3/s^2]$$

Using ε, ν to non-dimensionalize E and k:

$$E(k) = (\varepsilon \nu^5)^{1/4} \varphi(k\eta) = \eta {u_\eta}^2 \varphi(k\eta)$$

Where $u_\eta = v_d$ and $\varphi(k\eta) =$ *Kolmogorov spectrum function*.

Alternatively, using ε, k to non-dimensionalize E:

$$E(k) = \varepsilon^{2/3} k^{-5/3} \Psi(k\eta)$$

Where $\Psi(k\eta) =$ *compensated Kolmogorov spectrum function*.

$$\Psi(k\eta) = (k\eta)^{5/3} \varphi(k\eta)$$

And $k\eta > 2\pi\eta/l_e$.

2. **Kolmogorov's second similarity hypothesis**
   In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale l in the range $l_d < l < l_e$ have a universal form that is uniquely determined by ε, independent of ν.

$$\eta \ll l \ll l_0$$
$$l_d < l < l_e \sim \frac{1}{6} l_0$$

or:

$$1 \gg k\eta \gg \eta / l_0$$

$$k_d \eta = \frac{2\pi\eta}{l_d} > k\eta > \frac{2\pi\eta}{l_e} = k_e \eta$$

In the inertial subrange $E(k) = f(\varepsilon)$ only; thus, for $k\eta \ll 1, \Psi$ becomes independent of $k\eta, i.e., =$ constant=C.

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$ Kolmogorov -5/3 spectrum

C~1.5 = Kolmogorov universal constant.

$E(k)$ is a power law spectrum $= CAk^{-p}$, where $p = 5/3,\ A = \varepsilon^{2/3}$.

Most turbulence data come from stationary single point time series, which are converted to spatial data using Taylor's frozen turbulence hypothesis to obtain one-dimensional spectra that can be related to the 3D spectra using theory of isotropic turbulence (tensors).

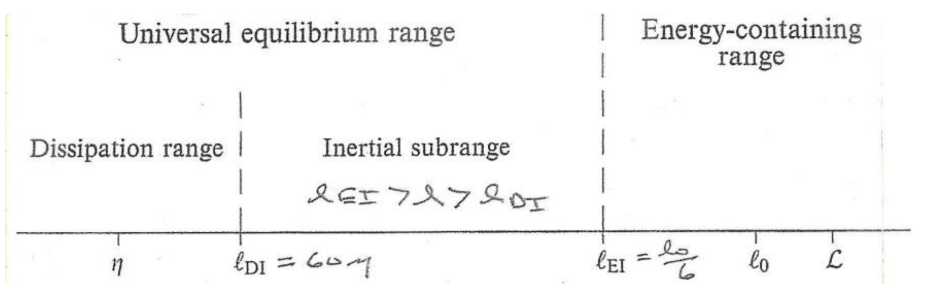

Fig. 6.1. Eddy sizes ℓ (on a logarithmic scale) at very high Reynolds number, showing the various lengthscales and ranges.

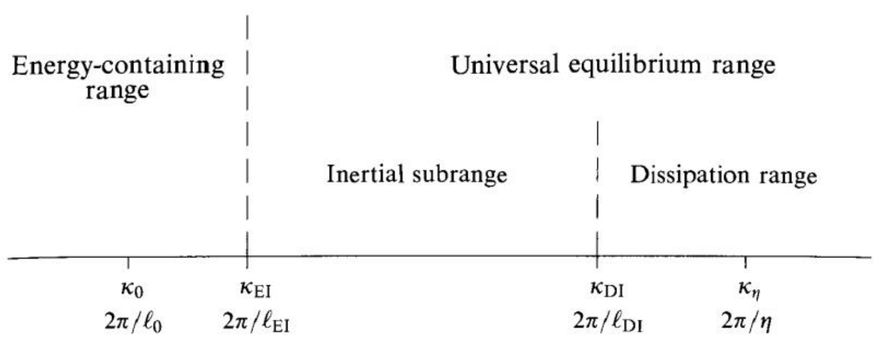

Fig. 6.12. Wavenumbers (on a logarithmic scale) at very high Reynolds number showing the various ranges.