Chapter 5 Dimensional Analysis and Modeling

The Need for Dimensional Analysis

Dimensional analysis is a process of formulating fluid mechanics problems in terms of nondimensional variables and parameters.

1. Reduction in Variables:

If $F(A_1, A_2, \ldots, A_n) = 0$,

Then $f(\Pi_1, \Pi_2, \ldots \Pi_{r<n}) = 0$

Thereby reduces number of experiments and/or simulations required to determine f vs. F

F = functional form
A_i = dimensional variables
Π_j = nondimensional parameters
$= \Pi_j(A_i)$
i.e., Π_j consists of nondimensional groupings of A_i's

2. Helps in understanding physics

3. Useful in data analysis and modeling

4. Fundamental to concept of similarity and model testing

   Enables scaling for different physical dimensions and fluid properties

Dimensions and Equations

Basic dimensions: F, L, and t or M, L, and t
F and M related by $F = Ma = MLT^{-2}$

The principle of homogeneity of dimensions is a rule that states that the dimensions of all terms in a physical expression should be the same. This principle is based on the fact that only physical quantities of the same kind can be added, subtracted, or compared. This principle is used to check the correctness and consistency of equations and mathematical relationships in various scientific fields.

Buckingham Π Theorem

In a physical problem including n dimensional variables in which there are m dimensions, the variables can be arranged into $r = n - \hat{m}$ independent nondimensional parameters Π_r (where usually $\hat{m} = m$).

$F(A_1, A_2, \ldots, A_n) = 0$

$f(\Pi_1, \Pi_2, \ldots \Pi_r) = 0$

A_i's = dimensional variables required to formulate problem (i = 1, n)

Π_j's = nondimensional parameters consisting of groupings of A_i's (j = 1, r)

F, f represents functional relationships between A_n's and Π_r's, respectively

$\hat{m}$ = rank of dimensional matrix
= m (i.e., number of dimensions) usually

Dimensional Analysis

Methods for determining Π_i's

1. Functional Relationship Method

Identify functional relationships $F(A_i)$ and $f(\Pi_j)$ by first determining A_i's and then evaluating Π_j's

a. Inspection	intuition
b. Step-by-step Method	text
c. Exponent Method	class

2. Nondimensionalize governing differential equations and initial and boundary conditions

Select appropriate quantities for nondimensionalizing the GDE, IC, and BC e.g. for M, L, and t

Put GDE, IC, and BC in nondimensional form

Identify Π_j's

Exponent Method for Determining Π_j's

1) determine the n essential quantities
2) select $\hat{m}$ of the A quantities, with different dimensions, that contain among them the $\hat{m}$ dimensions and use them as repeating variables together with one of the other A quantities to determine each Π.

For example, let A_1, A_2, and A_3 contain M, L, and t (not necessarily in each one, but collectively); then the Π_j parameters are formed as follows:

$$\left.\begin{aligned}
\Pi_1 &= A_1^{x_1} A_2^{y_1} A_3^{z_1} A_4 \\
\Pi_2 &= A_1^{x_2} A_2^{y_2} A_3^{z_2} A_5 \\
\Pi_{n-m} &= A_1^{x_{n-m}} A_2^{y_{n-m}} A_3^{z_{n-m}} A_n
\end{aligned}\right\}$$

Determine exponents such that Π_i's are dimensionless

3 equations and 3 unknowns for each Π_i

In these equations the exponents are determined so that each Π is dimensionless. This is accomplished by substituting the dimensions for each of the A_i in the equations and equating the sum of the exponents of M, L, and t each to zero. This produces three equations in three unknowns (x, y, t) for each Π parameter.

In using the above method, the designation of $\hat{m}$ = m as the number of basic dimensions needed to express the n variables dimensionally is not always correct. The correct value for $\hat{m}$ is the rank of the dimensional matrix, i.e., the next smaller square subgroup with a nonzero determinant.

Dimensional matrix =

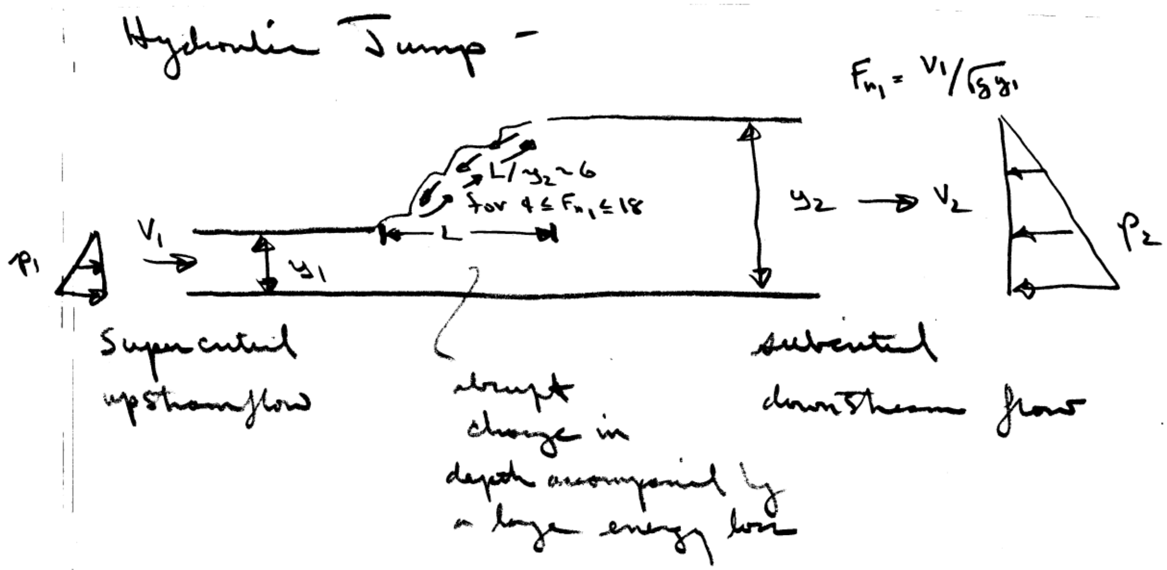
$$\begin{array}{c c}
 & \begin{array}{ccc} A_1 & \cdots & A_n \end{array} \\
\begin{array}{c} M \\ L \\ t \\ \\ \\ \\ \\ \\ \end{array} &
\left[\begin{array}{ccc}
a_{11} & \cdots & a_{1n} \\
 & & \\
a_{31} & \cdots & a_{3n} \\
o & \cdots & o \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\vdots & & \vdots \\
o & \cdots & o
\end{array}\right]
\end{array}$$

a_{ij} = exponent of M, L, or t in A_i

n x n matrix

Rank of dimensional matrix equals size of next smaller sub-group with nonzero determinant

Example: Hydraulic jump



HGL=p/γ+z; EGL=HGL+$\alpha V^2/2g$; $EGL_1=EGL_2+h_L$
for $h_t=h_p=0$

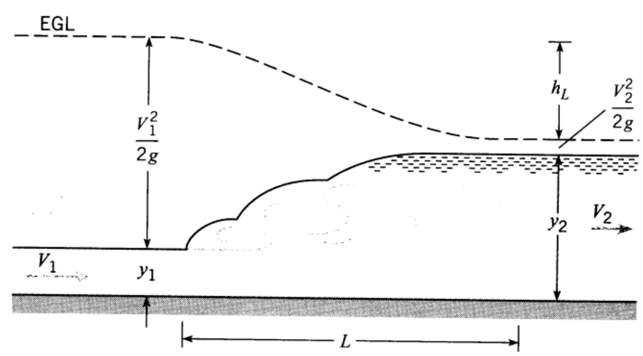


FIGURE 15.17
Definition sketch for the hydraulic jump.

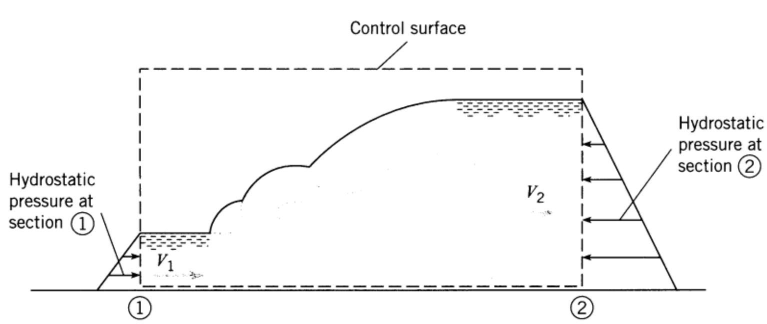


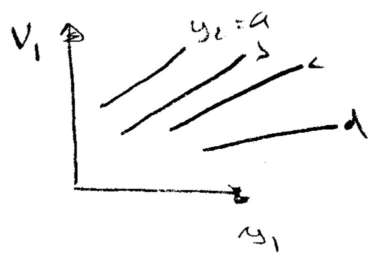
FIGURE 15.18
Control-volume analysis for the hydraulic jump.

Say we assume that

$$V_1 = V_1(\gamma, \mu, y_1, y_2) \quad \text{or } V_2 = V_1 y_1 / y_2$$

$$\boxed{\gamma = \rho g}$$

Dimensional analysis is a procedure whereby the functional relationship can be expressed in terms of r nondimensional parameters in which r < n = number of variables. Such a reduction is significant since in an experimental or numerical investigation a reduced number of experiments or calculations is extremely beneficial



1) γ, y_2 fixed; vary μ
2) γ, μ fixed; vary y_2
3) y_2, μ fixed; vary γ

Represents many, many experiments

In general: $F(A_1, A_2, \ldots, A_n) = 0$ dimensional form

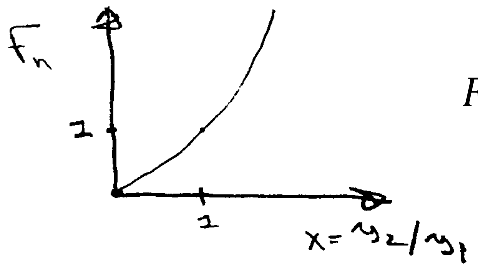
$f(\Pi_1, \Pi_2, \ldots \Pi_r) = 0$ nondimensional form with reduced # of variables

or $\Pi_1 = \Pi_1(\Pi_2, \ldots, \Pi_r)$

It can be shown that

$$F_r = \frac{V_1}{\sqrt{g y_1}} = F_r\left(\frac{y_2}{y_1}\right)$$

neglect μ (ρ drops out as will be shown) thus, only need one experiment to determine the functional relationship



$$F_r = \left[\frac{1}{2}x(1+x)\right]^{1/2}$$

X	F_r
0	0
½	.61
1	1
2	1.7
5	3.9

For this particular application we can determine the functional relationship through the use of a control volume analysis: (neglecting μ and bottom friction)

x-momentum equation: $\sum F_x = \sum V_x \rho \underline{V} \cdot \underline{A}$

$$\gamma\frac{y_1^2}{2} - \gamma\frac{y_2^2}{2} = V_1\rho(-V_1 y_1) + V_2\rho(V_2 y_2)$$

$$\frac{\gamma}{2}\left(y_1^2 - y_2^2\right) = \frac{\gamma}{g}\left(V_2^2 y_2 - V_1^2 y_1\right)$$

Note: each term in equation must have some units: principle of dimensional homogeneity, i.e., in this case, force per unit width N/m

continuity equation: $V_1 y_1 = V_2 y_2$

$$V_2 = \frac{V_1 y_1}{y_2}$$

$$\underbrace{\frac{\gamma y_1^2}{2}\left[1 - \left(\frac{y_2}{y_1}\right)^2\right]} = \underbrace{V_1^2 \frac{\gamma}{g} y_1\left(\frac{y_1}{y_2} - 1\right)}$$

pressure forces = inertial forces
due to gravity

now divide equation by $\dfrac{\left(1-\dfrac{y_2}{y_1}\right)y_1^3}{gy_2}$

$$\frac{V_1^2}{gy_1} = \frac{1}{2}\frac{y_2}{y_1}\left(1+\frac{y_2}{y_1}\right) \longleftarrow \text{dimensionless equation}$$

ratio of inertia forces/gravity forces = (Froude number)^2

note $\quad F_r = F_r(y_2/y_1)$ $\quad$ do not need to know both y_2 and y_1, only ratio to get F_r

Also, shows in an experiment it is not necessary to vary γ, y_1, y_2, V_1, and V_2, but only F_r and y_2/y_1

Next, can get an estimate of h_L from the energy equation (along free surface from $1\rightarrow2$)

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h_L$$

$$h_L = \frac{(y_2 - y_1)^3}{4y_1y_2}$$

$\neq f(\mu)$ due to assumptions made in deriving 1-D steady flow energy equations

Exponent method to determine Π_j's for Hydraulic jump

$F(g, V_1, y_1, y_2, \rho, \mu) = 0$ $\quad$ n = 6

$\frac{L}{T^2} \; \frac{L}{T} \; L \; L \; \frac{M}{L^3} \; \frac{M}{LT}$

use $V = V_1$, y_1, ρ as repeating variables

m = 3 $\Rightarrow$ r = n – m = 3

Assume $\hat{m} = m$ to avoid evaluating rank of 6 x 6 dimensional matrix

$\Pi_1 = V^{x1}\, y_1^{y1}\, \rho^{z1}\, \mu$

$= (LT^{-1})^{x1} (L)^{y1} (ML^{-3})^{z1} ML^{-1}T^{-1}$

L $\quad x_1 + y_1 - 3z_1 - 1 = 0 \qquad y_1 = 3z_1 + 1 - x_1 = -1$

T $\quad -x_1 \qquad\qquad - 1 = 0 \qquad x_1 = -1$

M $\quad z_1 \qquad\qquad + 1 = 0 \qquad z_1 = -1$

$$\Pi_1 = \frac{\mu}{\rho y_1 V} \quad \text{or} \quad \Pi_1^{-1} = \frac{\rho y_1 V}{\mu} = \text{Reynolds number} = \text{Re}$$

$\Pi_2 = V^{x2}\, y_1^{y2}\, \rho^{z2}\, g$

$= (LT^{-1})^{x2} (L)^{y2} (ML^{-3})^{z2} LT^{-2}$

L $\quad x_2 + y_2 - 3z_2 + 1 = 0 \qquad y_2 = -1 - x_2 = 1$

T $\quad -x_2 \qquad\qquad - 2 = 0 \qquad x_2 = -2$

M $\quad z_2 = 0$

$$\Pi_2 = V^{-2} y_1 g = \frac{g y_1}{V^2} \qquad \Pi_2^{-1/2} = \frac{V}{\sqrt{g y_1}} = \text{Froude number}$$

= Fr

$\Pi_3 = (LT^{-1})^{x3} (L)^{y3} (ML^{-3})^{z3}\, y_2$

L $\quad x_3 + y_3 + 3z_3 + 1 = 0 \qquad y_3 = -1$

T $\quad -x_3 = 0$

M $\quad -3z_3 = 0$

$$\Pi_3 = \frac{y_2}{y_1} \qquad \Pi_3^{-1} = \frac{y_1}{y_2} = \text{depth ratio}$$

$f(\Pi_1, \Pi_2, \Pi_3) = 0$

or $\quad \Pi_2 = \Pi_2(\Pi_1, \Pi_3)$

i.e., $F_r = F_r(Re, y_2/y_1)$
if we neglect μ then Re drops out

$$F_r = \frac{V_1}{\sqrt{gy_1}} = f\left(\frac{y_2}{y_1}\right)$$

Note that dimensional analysis does not provide the actual functional relationship. Recall that previously we used control volume analysis to derive

$$\frac{V_1^2}{gy_1} = \frac{1}{2}\frac{y_2}{y_1}\left(1+\frac{y_2}{y_1}\right)$$

the actual relationship between F vs. y_2/y_1

$F = F(Re, F_r, y_1/y_2)$
or $F_r = F_r(Re, y_1/y_2)$

dimensional matrix:

$$\begin{array}{c c} & \begin{array}{cccccc} g & V_1 & y_1 & y_2 & \rho & \mu \end{array} \\ \begin{array}{c} M \\ L \\ t \\ \\ \\ \\ \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & -1 \\ -2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Size of next smaller subgroup with nonzero determinant = 3 = rank of matrix

Example: Derivation of Kolmogorov Scales Using Dimensional Analysis

Nomenclature

l_0 ---- length scales of the largest eddies

η ---- length scales of the smallest eddies (Kolmogorov scale)

u_0 ---- velocity associated with the largest eddies

u_η ---- velocity associated with the smallest eddies

τ_0 ---- time scales of the largest eddies

τ_η ---- time scales of the smallest eddies

Assumptions:

1. For large Reynolds numbers, the small-scales of motion (small eddies) are statistically steady, isotropic (no sense of directionality), and independent of the detailed structure of the large-scales of motion.

2. **Kolmogorov's (1941) universal equilibrium theory**: The large eddies are not affected by viscous dissipation, but transfer energy to smaller eddies by inertial forces. The range of scales of motion where the dissipation in negligible is the inertial subrange.

3. **Kolmogorov's first similarity hypothesis**. In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions have a universal form that is uniquely determined by viscosity ν and dissipation rate ε.

Facts and Mathematical Interpretation:

Fact 1. Dissipation of energy through the action of molecular viscosity occurs at the smallest eddies, i.e., Kolmogorov scales of motion η. The Reynolds number (Re_η) of these scales are of order (1).

Fact 2. EFD confirms that most eddies break-up on a timescale of their turn-over time, where the turnover time depends on the local velocity and length scales. Thus, at Kolmogorov scale $\eta/u_\eta = \tau_\eta$.

Fact 3. The rate of dissipation of energy at the smallest scale is,

$$\varepsilon \equiv \nu S_{ij} S_{ij} \tag{1}$$

where $S_{ij} = \frac{1}{2}\left(\frac{\partial u_{\eta,i}}{\partial x_j} + \frac{\partial u_{\eta,j}}{\partial x_i}\right)$ is the rate of strain associated with the smallest eddies, $S_{ij} \equiv u_\eta/\eta$. Which yields,

$$\varepsilon \equiv \nu\left(u_\eta^2/\eta^2\right) \tag{2}$$

Fact 4. Kolmogorov scales of motion η, u_η, τ_η can be expressed as a function of ν, ε only.

Derivation:

Based on **Kolmogorov's first similarity hypothesis**, the small scales of motion are function of $F(\eta, u_\eta, \tau_\eta, \nu, \varepsilon)$ and determined by ν and ε only. Thus, ν and ε are repeating variables. The dimensions for ν and ε are L^2T^{-1} and L^2T^{-3}, respectively.

Herein, the exponential method is used:

$$F\left(\underset{L}{\eta}, \underset{\frac{L}{T}}{u_\eta}, \underset{T}{\tau_\eta}, \underset{\frac{L^2}{T}}{\nu}, \underset{\frac{L^2}{T^3}}{\varepsilon}\right)=0 \qquad n=5 \tag{3}$$

use ν and ε as repeating variables, $m=2 \Rightarrow r=n-m=3$

$$\Pi_1 = \nu^{x_1}\varepsilon^{y_1}\eta$$

$$=\left(L^2T^{-1}\right)^{x_1}\left(L^2T^{-3}\right)^{y_1}L$$

(4)

$$L \qquad 2x_1+2y_1+1=0$$

$$T \qquad -x_1-3y_1=0 \tag{5}$$

x_1=-3/4 and y_1=1/4

$$\Pi_1 = \eta\left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \tag{6}$$

$$\Pi_2 = \nu^{x_2}\varepsilon^{y_2}u_\eta$$

$$=\left(L^2T^{-1}\right)^{x_2}\left(L^2T^{-3}\right)^{y_2}\left(LT^{-1}\right) \tag{7}$$

$$L \qquad 2x_2+2y_2+1=0$$

$$T \qquad -x_2-3y_2-1=0 \tag{8}$$

x_2= y_2=-1/4

$$\Pi_2 = u_\eta/(\varepsilon\nu)^{1/4} \tag{9}$$

$$\Pi_3 = \nu^{x_3}\varepsilon^{y_3}\tau_\eta$$

$$=\left(L^2T^{-1}\right)^{x_3}\left(L^2T^{-3}\right)^{y_3}\left(T\right) \tag{10}$$

$$\begin{array}{ll} L & 2x_3 + 2y_3 = 0 \\ T & -x_3 - 3y_3 + 1 = 0 \end{array} \tag{11}$$

x_3=-1/2 and y_3=1/2

$$\Pi_3 = \tau_\eta \left(\frac{\varepsilon}{\nu}\right)^{1/2} \tag{12}$$

Analysis of the $\prod$ parameters give,

$$\Pi_1 \times \Pi_2 = \frac{u_\eta \eta}{\nu} = Re_\eta \equiv 1 \Rightarrow \text{Fact 1} \tag{13}$$

$$\frac{\Pi_2}{\Pi_1} \times \Pi_3 = \frac{u_\eta}{\eta}\tau_\eta = 1 \qquad \Rightarrow \text{Fact 2} \tag{14}$$

$$\frac{\Pi_2}{\Pi_1} = \frac{u_\eta}{\eta}\left(\frac{\varepsilon}{\nu}\right)^{1/2} \equiv 1 \qquad \Rightarrow \text{Fact 3} \tag{15}$$

$$\xrightarrow{\text{yields}} \quad \Pi_1 = \Pi_2 = \Pi_3 \equiv 1$$

Thus, Kolmogorov scales are:

$$\boxed{\begin{aligned} \eta &\equiv \left(\nu^3/\varepsilon\right)^{1/4}, \\ u_\eta &\equiv \left(\varepsilon\nu\right)^{1/4}, \\ \tau_\eta &\equiv \left(\nu/\varepsilon\right)^{1/2} \end{aligned}} \qquad \Rightarrow \text{Fact 4} \tag{16}$$

Ratios of the smallest to largest scales:

Based on Fact 2, the rate at which energy (per unit mass) is passed down the energy cascade from the largest eddies is,

$$\prod \equiv u_0^2 / \left(l_0 / u_0\right) = u_0^3 / l_0 \tag{17}$$

Based on Kolmogorov's universal equilibrium theory,

$$\varepsilon = u_0^3 / l_0 \equiv \nu \left(u_\eta^2 / \eta^2 \right)$$

(18)

Replace ε in Eqn. (16) using Eqn. (18) and note $\tau_0 = l_0 / u_0$,

$$\eta / l_0 \equiv \mathrm{Re}^{-3/4},$$
$$u_\eta / u_0 \equiv \mathrm{Re}^{-1/4},$$
$$\tau_\eta / \tau_0 \equiv \mathrm{Re}^{-1/2}$$

(19)

where $\mathrm{Re} = u_0 l_0 / \nu$

How large is η?

Cases	Re	η / l_o	l_o	η
Educational experiments	10^3	5.6×10^{-3}	~ 1 cm	5.6×10^{-3} cm
Model-scale experiments	10^6	3.2×10^{-5}	~ 3 m	9.5×10^{-5} m
Full-scale experiments	10^9	1.8×10^{-7}	~ 100 m	1.8×10^{-5} m

Much of the energy in this flow is dissipated in eddies which are less than fraction of a millimeter in size!!

Common Dimensionless Parameters for Fluid Flow Problems

Parameter	Definition	Qualitative ratio of effects	Importance
Reynolds number	$\mathrm{Re} = \frac{\rho U L}{\mu}$	$\frac{\text{Inertia}}{\text{Viscosity}}$	Almost always
Mach number	$\mathrm{Ma} = \frac{U}{a}$	$\frac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude number	$\mathrm{Fr} = \frac{U^2}{gL}$	$\frac{\text{Inertia}}{\text{Gravity}}$	Free-surface flow
Weber number	$\mathrm{We} = \frac{\rho U^2 L}{\Upsilon}$	$\frac{\text{Inertia}}{\text{Surface tension}}$	Free-surface flow
Rossby number	$\mathrm{Ro} = \frac{U}{\Omega_{earth} L}$	$\frac{\text{Flow velocity}}{\text{Coriolis effect}}$	Geophysical flows
Cavitation number (Euler number)	$\mathrm{Ca} = \frac{p - p_v}{\frac{1}{2}\rho U^2}$	$\frac{\text{Pressure}}{\text{Inertia}}$	Cavitation
Prandtl number	$\mathrm{Pr} = \frac{\mu c_p}{k}$	$\frac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$\mathrm{Ec} = \frac{U^2}{c_p T_0}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific-heat ratio	$k = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Strouhal number	$\mathrm{St} = \frac{\omega L}{U}$	$\frac{\text{Oscillation}}{\text{Mean speed}}$	Oscillating flow
Roughness ratio	$\frac{\varepsilon}{L}$	$\frac{\text{Wall roughness}}{\text{Body length}}$	Turbulent, rough walls
Grashof number	$\mathrm{Gr} = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Rayleigh number	$\mathrm{Ra} = \frac{\beta \Delta T g L^3 \rho^2 c_p}{\mu k}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\frac{T_w}{T_0}$	$\frac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aerodynamics, hydrodynamics
Lift coefficient	$C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Drag coefficient	$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Friction factor	$f = \frac{h_f}{(V^2/2g)(L/d)}$	$\frac{\text{Friction head loss}}{\text{Velocity head}}$	Pipe flow
Skin friction coefficient	$c_f = \frac{\tau_{wall}}{\rho V^2/2}$	$\frac{\text{Wall shear stress}}{\text{Dynamic pressure}}$	Boundary layer flow

Nondimensionalization of the Basic Equation

It is very useful and instructive to nondimensionalize the basic equations and boundary conditions. Consider the situation for ρ and μ constant and for flow with a free surface

Continuity: $\nabla \cdot \underline{V} = 0$

Momentum: $\rho \frac{D\underline{V}}{Dt} = -\nabla(p + \gamma z) + \mu \nabla^2 \underline{V}$

ρg = specific weight

Boundary Conditions:

1) fixed solid surface: $\underline{V} = 0$
2) inlet or outlet: $\underline{V} = \underline{V}_o$ $p = p_o$
3) free surface: $w = \frac{\partial \eta}{\partial t}$ $p = p_a - \gamma\left(R_x^{-1} + R_y^{-1}\right)$

$(z = \eta)$ surface tension

All variables are now nondimensionalized in terms of ρ and

U = reference velocity

L = reference length

$\underline{V}^* = \frac{\underline{V}}{U}$ $t^* = \frac{tU}{L}$

$\underline{x}^* = \frac{\underline{x}}{L}$ $p^* = \frac{p + \rho g z}{\rho U^2}$

All equations can be put in nondimensional form by making the substitution

$$\underline{V} = \underline{V}^* U$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} = \frac{U}{L}\frac{\partial}{\partial t^*}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\hat{i} + \frac{\partial}{\partial y^*}\frac{\partial y^*}{\partial y}\hat{j} + \frac{\partial}{\partial z^*}\frac{\partial z^*}{\partial z}\hat{k}$$

$$= \frac{1}{L}\nabla^*$$

and $\frac{\partial u}{\partial x} = \frac{1}{L}\frac{\partial}{x^*}\left(U u^*\right) = \frac{U}{L}\frac{\partial u^*}{\partial x^*}$ etc.

Result: $\nabla^* \cdot \underline{V}^* = 0$

$$\frac{D\underline{V}^*}{Dt} = -\nabla^* p^* + \underbrace{\frac{\mu}{\rho V L}}_{Re^{-1}}\nabla^{*2}\underline{V}^*$$

1) $\underline{V}^* = 0$

2) $\underline{V}^* = \frac{\underline{V}_o}{U}$ $\qquad p^* = \frac{p_o}{\rho V^2}$

3) $w^* = \frac{\partial \eta^*}{\partial t^*}$

$$p^* = \underbrace{\frac{p_o}{\rho V^2}}_{\text{Pressure coefficient}} + \underbrace{\frac{gL}{U^2}}_{Fr^{-2}} z^* + \underbrace{\frac{\gamma}{\rho V^2 L}}_{We^{-1}}\left(R_x^{*-1} + R_y^{*-1}\right)$$

V = U

Similarity and Model Testing

Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for model and prototype

$$\Pi_{i\ \text{model}} = \Pi_{i\ \text{prototype}} \qquad i = 1, r = n - \hat{m}\ (m)$$

Enables extrapolation from model to full scale

However, complete similarity usually not possible

Therefore, often it is necessary to use Re, or Fr, or Ma scaling, i.e., select most important Π and accommodate others as best possible

Types of Similarity:

1) Geometric Similarity (similar length scales):
A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratios

$$\alpha = L_m/L_p \qquad (\alpha < 1)$$

1/10 or 1/50

2) Kinematic Similarity (similar length and time scales):
The motions of two systems are kinematically similar if homologous (same relative position) particles lie at homologous points at homologous times

3) Dynamic Similarity (similar length, time and force (or mass) scales):
   in addition to the requirements for kinematic similarity the model and prototype forces must be in a constant ratio

<u>Model Testing in Water</u> (with a free surface)

$$F(D, L, V, g, \rho, \nu) = 0$$

n = 6 and m = 3 thus r = n - m = 3 pi terms

In a dimensionless form,

$$f(C_D, Fr, Re) = 0$$

or

$$C_D = f(Fr, Re)$$

where

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 L^2}$$

$$Fr = \frac{V}{\sqrt{gL}}$$

$$Re = \frac{VL}{\nu}$$

If $Fr_m = Fr_p$ or $\frac{V_m}{\sqrt{gL_m}} = \frac{V_p}{\sqrt{gL_p}}$

$$V_m = \frac{\sqrt{gL_m}}{\sqrt{gL_p}} V_p = \sqrt{\alpha} V_p \qquad \text{Froude scaling}$$

and $Re_m = Re_p$ or $\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$

$$\frac{\nu_m}{\nu_p} = \frac{V_m L_m}{V_p L_p} = \alpha^{1/2}\alpha = \alpha^{3/2}$$

Then,

$$C_{D_m} = C_{D_p} \text{ or } \frac{D_m}{\rho_m V_m^2 L_m^2} = \frac{D_p}{\rho_p V_p^2 L_p^2}$$

However, impossible to achieve, since

if $\alpha = 1/10$, $\nu_m = 3.1\times10^{-8}\, m^2/s < 1.2\times10^{-7}\, m^2/s$

For mercury $\nu = 1.2\times10^{-7}\, m^2/s$

Alternatively, one could maintain Re similarity and obtain

$$V_m = V_p/\alpha$$

But if $\alpha = 1/10$, $V_m = 10V_p$,

High speed testing is difficult and expensive.

$$\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p}$$

$$\frac{g_m}{g_p} = \frac{V_m^2}{V_p^2}\frac{L_p}{L_m}$$

$$\frac{g_m}{g_p} = \frac{V_m^2}{V_p^2}\frac{L_p}{L_m}$$

$$\frac{g_m}{g_p} = \frac{1}{\alpha^2} \times \frac{1}{\alpha} = \alpha^{-3}$$

$$g_m = \frac{g_p}{\alpha^3}$$

But if $\alpha = 1/10$, $g_m = 1000 g_p$
Impossible to achieve

Model Testing in Air

$$F(D, L, V, \rho, v, a) = 0$$

n = 6 and m = 3 thus r = n - m = 3 pi terms

In a dimensionless form,

$$f(C_D, Ma, Re) = 0$$

or

$$C_D = f(Re, Ma)$$

where

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 L^2}$$

$$Re = \frac{VL}{v}$$

$$Ma = \frac{V}{a}$$

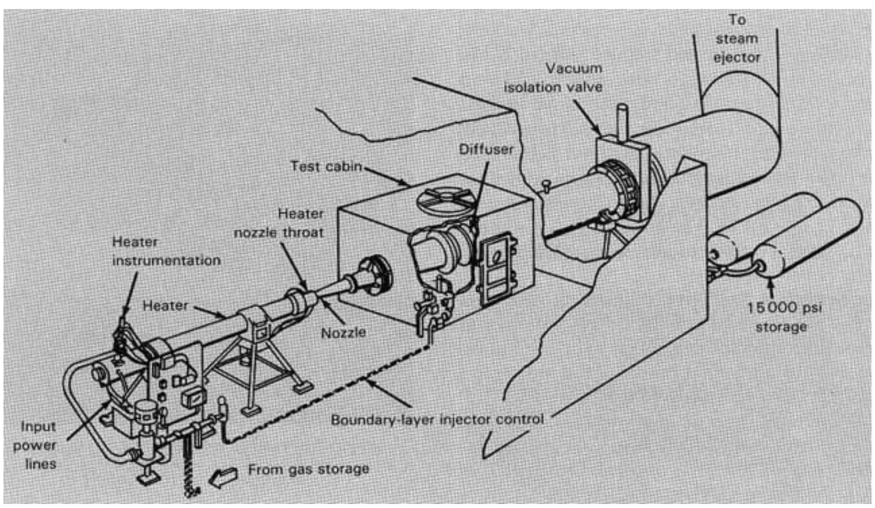
If $\dfrac{V_m L_m}{\nu_m} = \dfrac{V_p L_p}{\nu_p}$ and $\dfrac{V_m}{a_m} = \dfrac{V_p}{a_p}$

Then,

$$C_{D_m} = C_{D_p} \text{ or } \frac{D_m}{\rho_m V_m^2 L_m^2} = \frac{D_p}{\rho_p V_p^2 L_p^2}$$

However, $\dfrac{\nu_m}{\nu_p} = \dfrac{L_m}{L_p}\underbrace{\left[\dfrac{a_m}{a_p}\right]}_{1} = \alpha$

not easily achieved. Need fluid with high speed of sound and low viscosity.
https://history.nasa.gov/SP-440/ch6-15.htm



This helium blowdown tunnel at Ames attained Mach 50. Despite Its very low liquefaction point, the helium had to be heated to 1500 ° F to preclude any liquefaction during expansion.

Therefore, in wind tunnel testing Re scaling is also usually violated

In hydraulics model studies, Fr scaling used, but lack of We similarity can cause problems. Therefore, often models are distorted, i.e., vertical scale is increased by 10 or more compared to horizontal scale

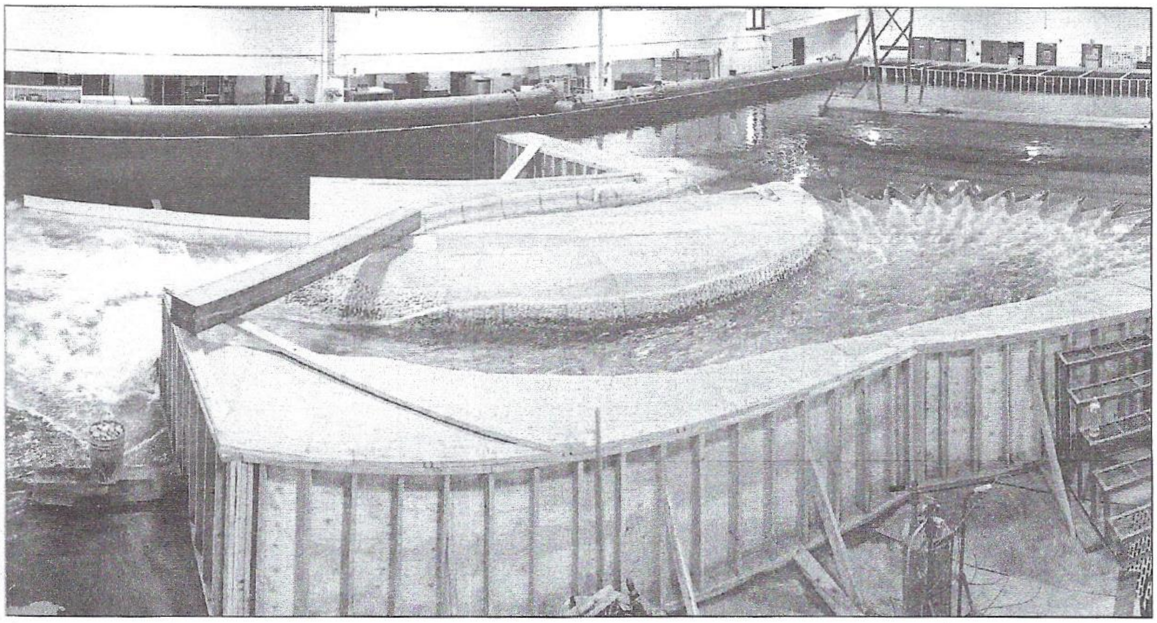


Fig. 5.8 Hydraulic model of the Isabella Lake Dam Safety Modification Project. The model scale is 1:45, and was built in 2014 at Utah State University's Water Research Laboratory. *(Courtesy of the U.S. Army photo by John Prettyman/Released.)*

Vertical scale distorted to avoid Weber number effects, i.e., horizontal scale is 1:1000 vs. vertical scale is 1:100; thus, model is deeper relative to its horizontal dimensions

Ship model testing:

$C_T = (Re, F_r) = C_w(F_r) + C_v(Re)$

V_m determined for F_r scaling

$C_{wm} = C_{Tm} - C_v$

$C_{Ts} = C_{wm} + C_v$

Based on flat plate of same surface area