Chapter 8: Channel and Pipe Flows (Chapter 7.1-7.2 Pope)

Part 1: Channel Flow

Internal flows: pipes, ducts, and turbomachinery

External flows: ships, aircrafts, road/rail vehicles

Environmental flows: atmospheric BL, rivers, and oceans

Canonical flows: fully developed channel and pipe flows and flat plate boundary layer. Former is parallel and latter nearly parallel.

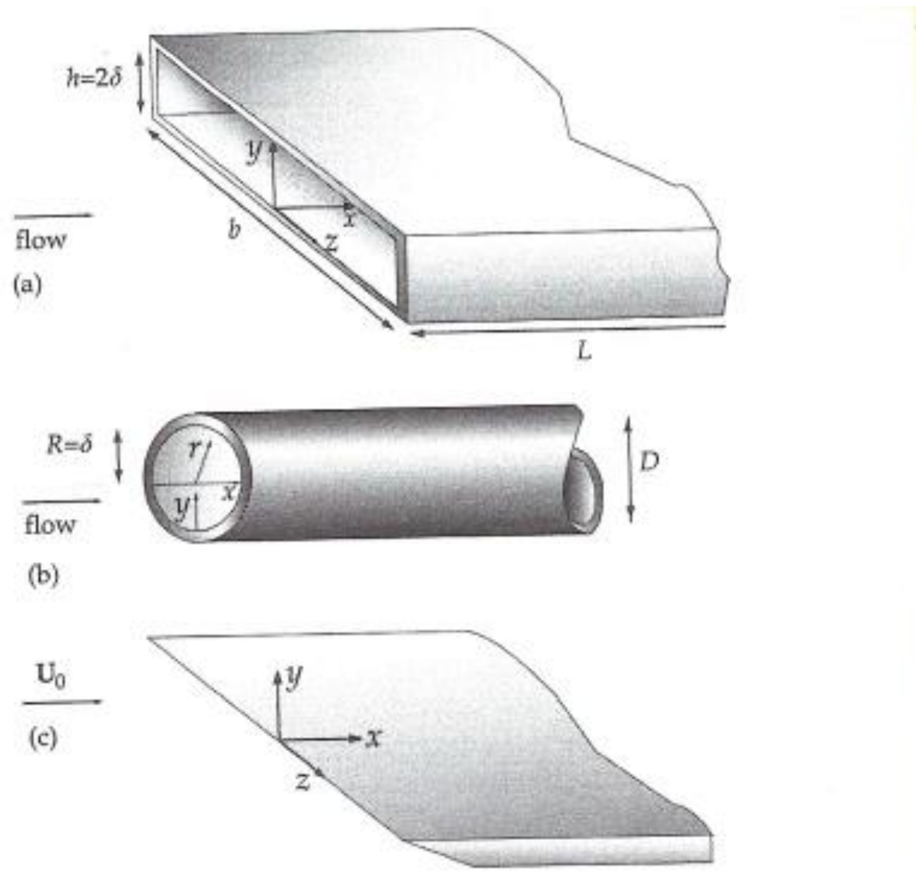


Fig. 7.1. Sketches of (a) channel flow, (b) pipe flow, and (c) a flat-plate boundary layer.

Focus: mean flow velocity profiles, friction laws, Reynolds stresses, and TKE budgets

Channel flow (Appendix: Laminar Flow Solution)

$h = 2\delta$, $L/\delta \gg 1$ long, $b/\delta \gg 1$ wide, and no cross flow $\langle W \rangle = 0$.

Large x distant from inlet: fully developed flow, statistically stationary and 1D, such that flow $f(y)$ and symmetric about $y = \delta =$ mid plane. Reynolds numbers used to characterize the flow are:

$$Re = \frac{\overline{U}(2\delta)}{\nu} \quad Re_0 = \frac{U_0 \delta}{\nu}$$

$$U_0 = \langle U \rangle_{y=\delta} \quad \text{centerline velocity}$$

$$\overline{U} = \frac{1}{\delta}\int_0^\delta \langle U \rangle dy \quad \text{average/bulk velocity}$$

Flow laminar for $Re < 1350$ and turbulent for $Re > 1800$, but transition effects up to $Re = 3000$.

Continuity:

$$\langle V \rangle_y = 0$$

Since $\langle U \rangle_x + \langle W \rangle_z = 0$ and with BCs $\langle V \rangle = 0$ at $y = 0$ and $y = 2\delta$,

$$\langle V \rangle = 0.$$

Streamwise momentum equation:

$$0 = -\frac{1}{\rho}\langle p \rangle_x + \nu \langle U \rangle_{yy} - \langle uv \rangle_y \quad (1)$$

Lateral momentum equation:

$$0 = -\frac{1}{\rho}\langle p \rangle_y - \langle v^2 \rangle_y \quad (2)$$

Integrating Eq. (2) across dy with limits 0 to y and using $\langle v^2 \rangle = 0$ at $y = 0$ gives:

$$\langle v^2 \rangle + \frac{\langle p \rangle}{\rho} = \frac{p_w(x)}{\rho}$$

Where $p_w(x) = \langle p(x), 0\rangle$ = mean pressure bottom wall. Differentiating with respect to x:

$$\frac{\partial\langle p\rangle}{\partial x} = constant = \frac{dp_w}{dx} \neq f(y)$$

Eq. (1) can be rewritten as:

$$\frac{d\tau}{dy} = \frac{dp_w}{dx} \qquad (3)$$

Where:

$$\tau(y) = \rho\nu\frac{d\langle U\rangle}{dy} - \rho\langle uv\rangle$$

represents the total shear stress. There is no acceleration and balance of forces between cross stream shear stress gradient and axial normal stress gradient.

$$\frac{d\tau}{dy} = \frac{dp_w}{dx} = \text{constant}$$

$\tau(y)$ anti-symmetric about mid plane ($y = \delta$): $\tau_w = \tau(0)$, $\tau_w = -\tau(2\delta)$, $0 = \tau(\delta)$. Therefore, solution of Eq. (3) is given by:

$$\tau(y) = \tau_w\left(1 - \frac{y}{\delta}\right)$$

Such that

$$-\frac{dp_w}{dx} = \frac{\tau_w}{\delta}$$

Skin friction coefficients:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} \qquad C_f = \frac{\tau_w}{\frac{1}{2}\rho \overline{U}^2}$$

Flow driven by pressure drop → in fully developed region $p_{w_x} < 0$ balanced by $\tau_y = -\tau_w/\delta$. Note that shear stress profile $\tau(y)$ is independent flow properties (ρ, ν) and state of fluid motion (i.e., laminar, or turbulent).

Near wall shear stress

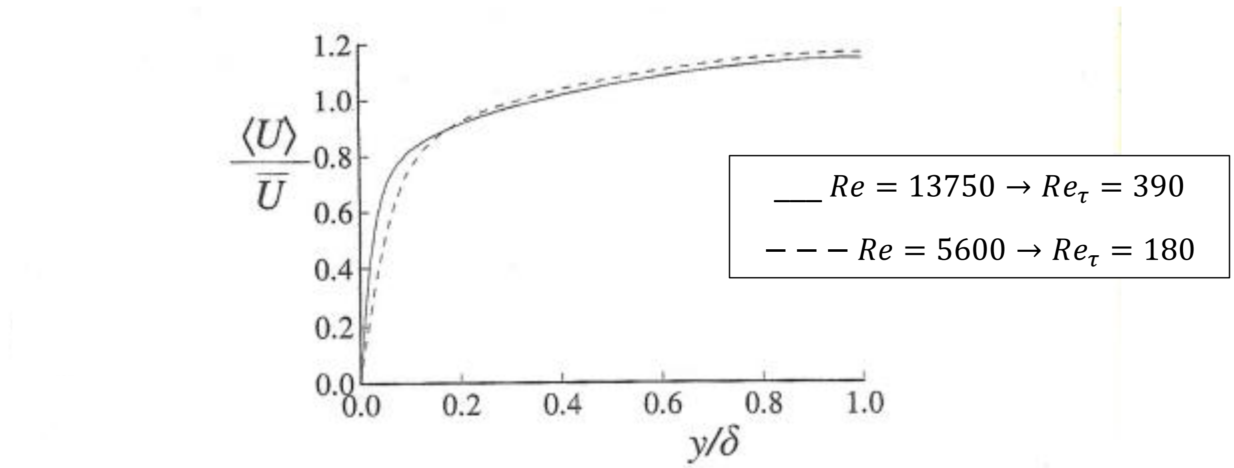


Fig. 7.2. Mean velocity profiles in fully developed turbulent channel flow from the DNS of Kim *et al.* (1987): dashed line, Re = 5,600; solid line, Re = 13,750.

$$\tau(y) = \rho\nu \frac{d\langle U \rangle}{dy} - \rho\langle uv \rangle$$

$$\tau(0) = \rho\nu \frac{d\langle U \rangle}{dy}\bigg|_0 = \tau_w$$

Since RS at $y = 0$ are zero.

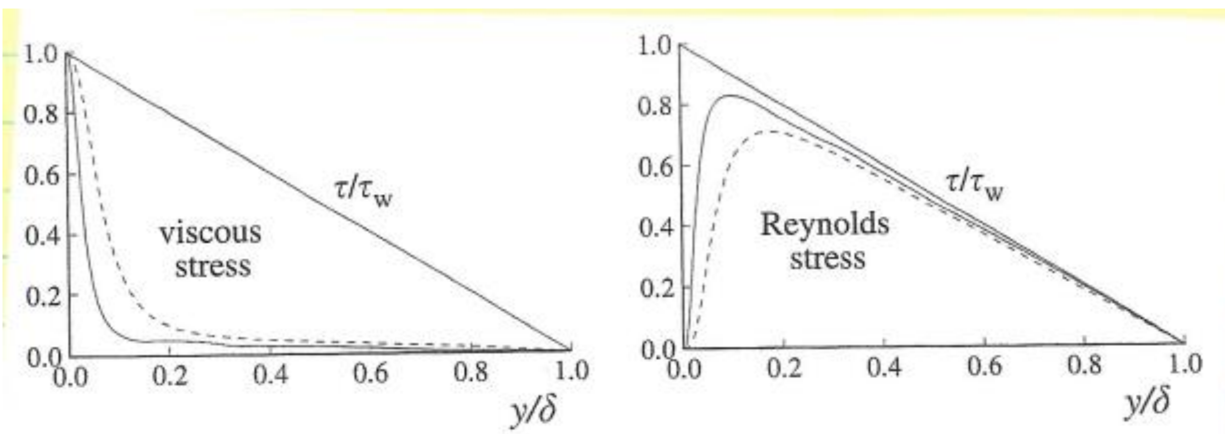


Fig. 7.3. Profiles of the viscous shear stress, and the Reynolds shear stress in turbulent channel flow: DNS data of Kim *et al.* (1987): dashed line, Re = 5,600; solid line, Re = 13,750.

Near wall viscous stress dominates vs. free shear flows where for high Re viscous stress negligible vs. RS.

Near the wall, the viscosity is influential → $\langle U \rangle = f(Re)$ in contrast to free shear flow.

Near wall: τ_w, ν, and ρ important and define:

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} \quad \text{friction velocity}$$

$$\delta_\nu = \nu\sqrt{\frac{\rho}{\tau_w}} = \frac{\nu}{u_\tau} \quad \text{viscous length scale}$$

Friction Reynolds number:

$$Re_\tau = \frac{u_\tau \delta}{\nu} = \frac{\delta}{\delta_\nu} \text{ ratio channel half height to viscous length scale}$$

Local Reynolds number:

$$y^+ = \frac{y}{\delta_\nu} = \frac{u_\tau y}{\nu}$$

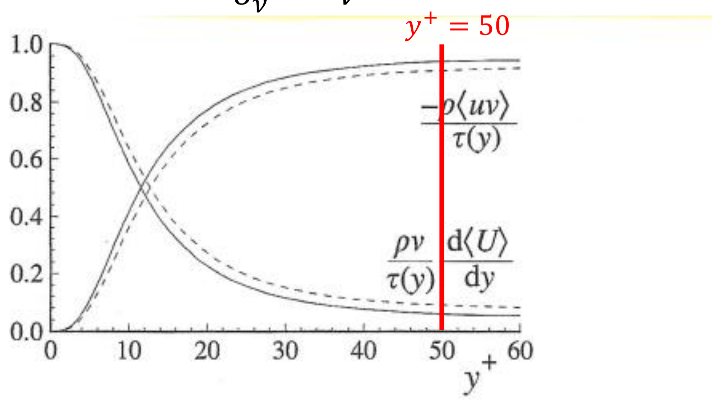


Fig. 7.4. Profiles of the fractional contributions of the viscous and Reynolds stresses to the total stress. DNS data of Kim *et al.* (1987): dashed lines, Re = 5,600; solid lines, Re = 13,750.

(Recall $-\langle uv \rangle$ nearly constant $y^+ \geq 50$ assumption used Bernard derive log law)

Note for different Re Fig. 7.4 results almost collapse when represented vs y^+; and

$$\frac{\mu\langle U\rangle_y}{\tau(y)} = \begin{cases} 100\% \;\; y^+ = 0 \\ 50\% \; y^+ = 12 \\ < 10\% \; y^+ = 50 \end{cases}$$

y^+ is used to define different near wall regions/layers.

1) $y^+ < 50$ viscous wall region $\rightarrow$ $\tau = f(\mu)$
2) $y^+ > 50$ outer layer $\rightarrow$ $\tau \neq f(\mu)$
3) $y^+ < 5$ viscous sublayer $\langle uv \rangle \ll \mu \langle U \rangle_y$

As Re increases, δ_ν/δ decreases, since $= Re_\tau^{-1}$.

Mean velocity profiles.

$$\tau_w = -\delta p_{w_x}$$

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{-\frac{\delta}{\rho} p_{w_x}}$$

Dimensional variables: ρ, ν, δ, and p_{w_x} $(or\ u_\tau)$ can form two non-dimensional groups, such that:

$$\frac{\langle U \rangle}{u_\tau} = f\left(\frac{y}{\delta}, Re_\tau\right)$$

Where f = universal non-dimensional function.

Similarly, for $\langle U \rangle_y$:

$$\langle U \rangle_y = \frac{u_\tau}{y} f\left(\frac{y}{\delta_\nu}, \frac{y}{\delta}\right)$$

$$= \frac{u_\tau}{y} f\left(y^+, \frac{y}{\delta}\right)$$

Idea is that δ_ν appropriate for $y^+ < 50$, while δ for $y^+ > 50$.

Note that:

$\left(\frac{y}{\delta_\nu}\right) / \left(\frac{y}{\delta}\right) = Re_\tau$ which shows that δ and δ_ν share same information as $\frac{y}{\delta}$ and Re_τ

Law of the wall (inner layer)

Prandtl postulated that at high Re, close to the wall $(y/\delta \ll 1)$, mean velocity profile depends on viscous scales:

$$(U)_y = \frac{u_\tau}{y}\Phi_I\left(\frac{y}{\delta_\nu} = y^+\right) \neq f(\delta, U_0) \quad (4)$$

Define

$$u^+ = \frac{\langle U \rangle}{u_\tau}$$

Such that Eq. (4) becomes:

$$\frac{du^+}{dy^+} = \frac{1}{y^+}\Phi_I(y^+) \quad (5)$$

Integrating Eq. (5) gives the law of the wall:

$$u^+ = f_w(y^+)$$

Where:

$$f_w(y^+) = \int_0^{y^+} \frac{1}{y^+}\Phi_I(y^+)dy^+$$

Is a universal function for channel flow, pipe, and BL flows, i.e., wall flows.

The viscous sublayer

$$u^+ = \frac{\langle U \rangle}{u_\tau} = f_w(y^+)$$

No-slip condition:

$$u^+(0) = f_w(0) = 0$$

Shear stress:

$$\tau_w = \rho\nu\left(\frac{d\langle U\rangle}{dy}\right)_{y=0}$$

Or equivalently, normalizing using viscous scales:

$$\frac{du^+}{dy^+}(0) = \frac{du^+}{dy}\frac{dy}{dy^+} = \frac{\langle U\rangle_y}{u_\tau}\frac{\nu}{u_\tau}$$

$$= \frac{\frac{\tau_w \nu}{\mu}}{\frac{\tau_w}{\rho}} = f_w'(0) = 1$$

Hence, Taylor-series expansion for $f_w(y^+)$ for small y^+ is:

$$f_w(y^+) = y^+ + O\left({y^+}^2\right) = u^+$$

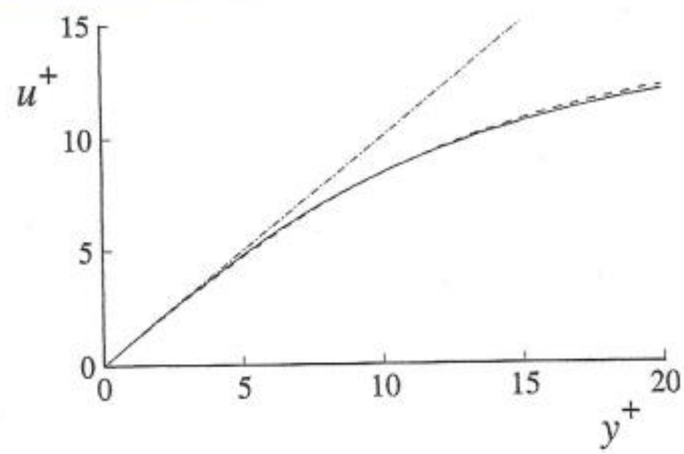


Fig. 7.5. Near-wall profiles of mean velocity from the DNS data of Kim *et al.* (1987): dashed line, Re = 5,600; solid line, Re = 13,750; dot–dashed line, $u^+ = y^+$.

Small departure from $u^+ = y^+$ for $y^+ < 5$, whereas significant (25%) for $y^+ >$ 12.

The Log Law

Inner layer usually defined as $\frac{y}{\delta} < 0.1$. At high Re, outer part of the inner layer corresponds to large $y^+ \sim 0.1\delta/\delta_\nu = 0.1Re_\tau \gg 1$.

In this region, viscosity has little effect → $\langle U \rangle \neq f(\nu)$

Therefore, $\Phi_I\left(\frac{y}{\delta_\nu}\right)$ in Eq. (4) becomes independent of δ_ν → constant:

$$\Phi_I(y^+) = \frac{1}{k}$$

For $y/\delta \ll 1$ and $y^+ \gg 1$.

Thus, in this region, the mean velocity gradient is:

$$\frac{du^+}{dy^+} = \frac{1}{ky^+}$$

Which integrates to:

$$u^+ = \frac{1}{k}\ln y^+ + B$$

With $k = 0.41$ and $B = 5.2$, “universal constants.”

Valid for $y^+ > 30$ except near δ (mid channel).

The region between viscous sublayer and log law region $(5 < y^+ < 30)$ is called the buffer layer: transition region between viscous and turbulence dominated regions where RS peaks.

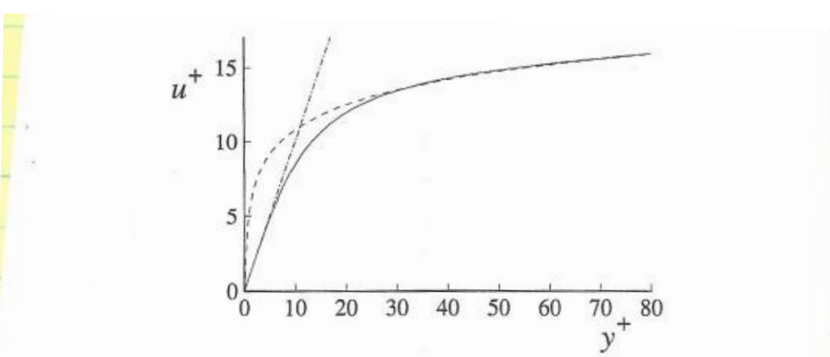


Fig. 7.6. Near-wall profiles of mean velocity: solid line, DNS data of Kim *et al.* (1987): Re = 13,750; dot–dashed line, $u^+ = y^+$; dashed line, the log law, Eqs. (7.43)–(7.44).

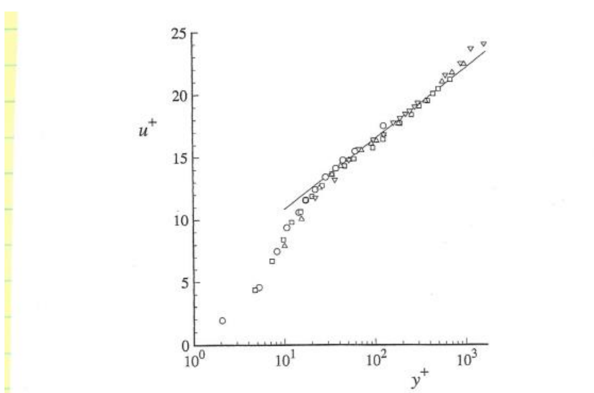


Fig. 7.7. Mean velocity profiles in fully developed turbulent channel flow measured by Wei and Willmarth (1989): ○, $Re_0 = 2{,}970$; □, $Re_0 = 14{,}914$; Δ, $Re_0 = 22{,}776$; ∇, $Re_0 = 39{,}582$; line, the log law, Eqs. (7.43)–(7.44).

The velocity defect law

Outer layer $y^+ > 50$: $\Phi(y/\delta_\nu, y/\delta) \neq f(\nu)$

$$\Phi\left(\frac{y}{\delta_\nu}, \frac{y}{\delta}\right) \rightarrow \Phi_o\left(\frac{y}{\delta}\right)$$

Therefore, Eq. (4) becomes:

$$(U)_y = \frac{u_\tau}{y}\Phi_o\left(\frac{y}{\delta}\right)$$

And integrating between y and δ yields the velocity defect law due to von Karman:

$$\frac{U_0 - \langle U\rangle}{u_\tau} = F_D\left(\frac{y}{\delta}\right)$$

where $U_0 - \langle U\rangle$ = difference between centerline and mean velocities and,

$$F_D\left(\frac{y}{\delta}\right) = \int_{\frac{y}{\delta}}^{1} \frac{1}{y'}\Phi_o(y')dy'$$

And F_D is different in different flows, i.e., not universal function like $f_w(y^+)$.

At sufficiently high Re (>20,000) there is an overlap region between inner layer $y/\delta < 0.1$) and outer layer ($y/\delta_\nu > 50$) where both

$$(U)_y = \frac{u_\tau}{y}\Phi_I\left(\frac{y}{\delta_\nu}\right)$$

And

$$(U)_y = \frac{u_\tau}{y}\Phi_o\left(\frac{y}{\delta}\right)$$

are valid, such that:

$$\frac{y}{u_\tau}(U)_y = \Phi_I\left(\frac{y}{\delta_\nu}\right) = \Phi_o\left(\frac{y}{\delta}\right)$$

For $\delta_\nu \ll y \ll \delta$.

This equation can be satisfied in the overlap region only by Φ_I and Φ_o being constant, i.e.,

$$\frac{y}{u_\tau}(U)_y = \frac{1}{k} \quad (\log \text{law})$$

This shows an alternative derivation of the log law and established the form of the velocity defect law for small y/δ:

$$\frac{U_0 - \langle U \rangle}{u_\tau} = F_D\left(\frac{y}{\delta}\right) = -\frac{1}{k}\ln\left(\frac{y}{\delta}\right) + B_1$$

Where B_1 is a flow dependent constant.

Let $U_{0,log}$ be the value of $\langle U \rangle$ on the centerline extrapolated by the log law, then:

$$B_1 = \frac{U_0 - U_{0,log}}{u_\tau} = F_D$$

DNS: $B_1 = 0.2$

Other measurements: $B_1 \sim 0.7$.

Larger for BL than channel and pipe flows.

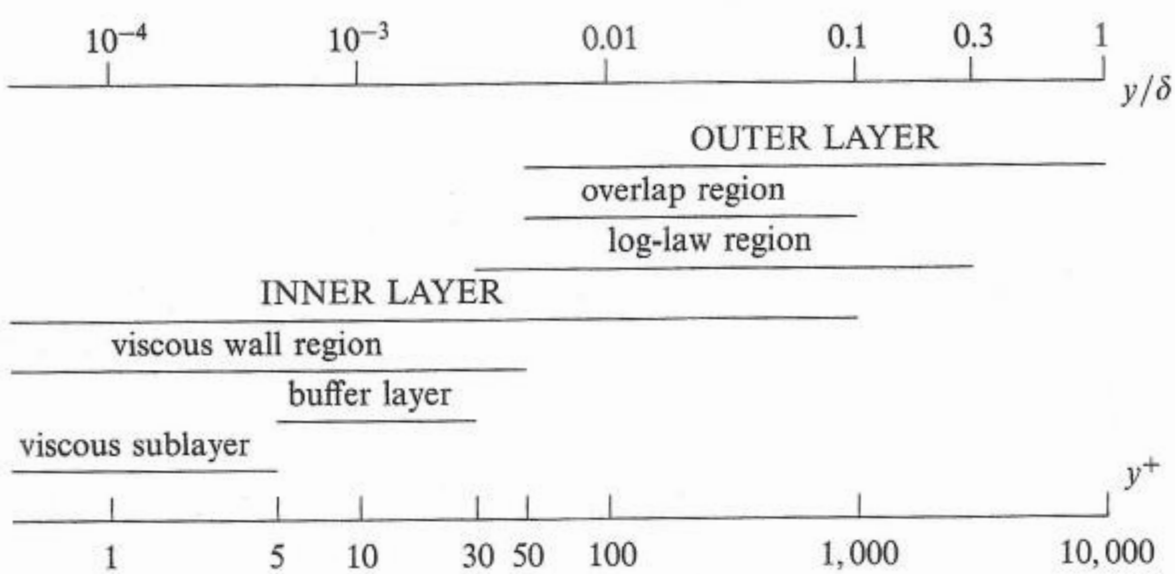


Fig. 7.8. A sketch showing the various wall regions and layers defined in terms of $y^+ = y/\delta_\nu$ and y/δ, for turbulent channel flow at high Reynolds number ($Re_\tau = 10^4$).

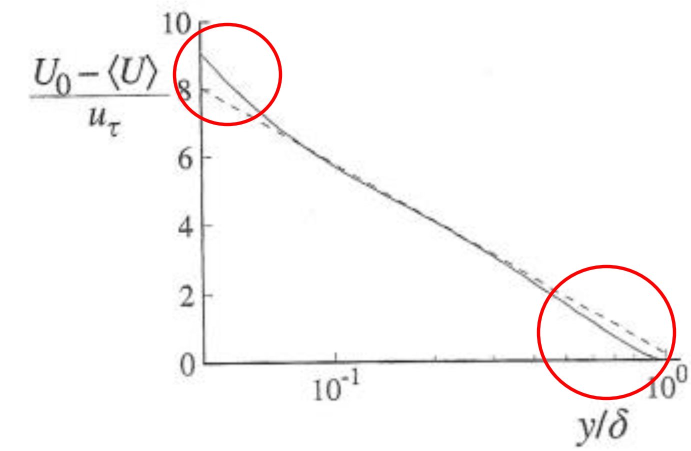
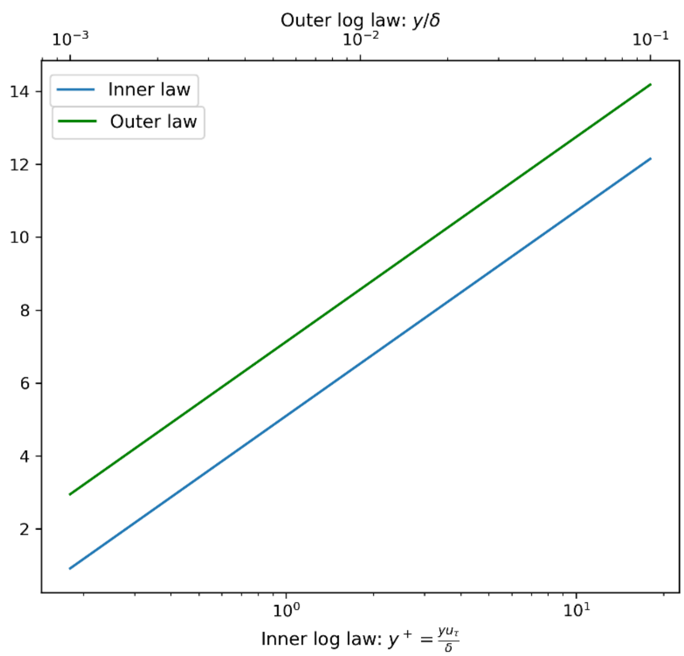


Fig. 7.9. The mean velocity defect in turbulent channel flow. Solid line, DNS of Kim *et al.* (1987), Re = 13, 750; dashed line, log law, Eqs. (7.43)–(7.44).



Inner log law:

$$\frac{\langle U \rangle}{u_\tau} = \frac{1}{k}\ln(y^+) + B \quad (k = 0.41, B = 5.1)$$

Outer log law:

$$\frac{U_0 - \langle U \rangle}{u_\tau} = -\frac{1}{k}\ln\left(\frac{y}{\delta}\right) + B_1 \rightarrow \frac{\langle U \rangle}{u_\tau} = \frac{U_0}{u_\tau} + \frac{1}{k}\ln\left(\frac{y}{\delta}\right) + B_1$$

$B_1 = 0.2, \qquad Re_\tau = 180, \qquad Re = 13750, \qquad U_0/u_\tau = 5\log_{10} Re = 4.14$

The friction law and the Reynolds number

An approximation for the bulk velocity can be obtained using the log law:

$$\frac{U_0 - \langle U \rangle}{u_\tau} = F_D\left(\frac{y}{\delta}\right) = -\frac{1}{k}\ln\left(\frac{y}{\delta}\right) + B_1$$

and assuming $B_1 = 0$, i.e., neglecting outer and inner layers, i.e., assume log law valid over entire channel.

$$\frac{U_0 - \overline{U}}{u_\tau} = \frac{1}{\delta}\int_0^\delta \frac{U_0 - \langle U \rangle}{u_\tau}\,dy$$

$$\approx \frac{1}{\delta}\int_0^\delta -\frac{1}{k}\ln\left(\frac{y}{\delta}\right)dy = \frac{1}{k}\sim 2.4 \quad (6)$$

DNS: 2.6, data: 2-3.

Log law in the inner layer:

$$\frac{\langle U \rangle}{u_\tau} = \frac{1}{k}\ln\left(\frac{y}{\delta_\nu}\right) + B$$

Whereas in the outer layer:

$$\frac{U_0 - \langle U \rangle}{u_\tau} = -\frac{1}{k}\ln\left(\frac{y}{\delta}\right) + B_1$$

Adding these two together such that f(y) vanishes:

$$\frac{U_0}{u_\tau} = \frac{1}{k}\ln\left(\frac{\delta}{\delta_\nu}\right) + B + B_1$$

$$\frac{U_0}{u_\tau} = \frac{1}{k}\ln\left[Re_0\left(\frac{U_0}{u_\tau}\right)^{-1}\right] + B + B_1 \quad (7)$$

$$\frac{\delta}{\delta_\nu} = Re_\tau$$

$$Re_0 = \frac{U_0\delta}{\nu}$$

For given Re_0, this equation can be solved for U_0/u_τ, i.e., center line velocity normalized u_τ, which provides:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} = 2\left(\frac{u_\tau}{U_0}\right)^2$$

Using Eq. (6):

$$\frac{U_0 - \overline{U}}{u_\tau} = \frac{1}{k} \rightarrow \overline{U} = u_\tau\left(\frac{U_0}{u_\tau} - \frac{1}{k}\right) \quad \text{bulk velocity}$$

$$Re = \frac{2\overline{U}\delta}{\nu} \qquad C_f = \frac{\tau_w}{\frac{1}{2}\rho \overline{U}^2}$$

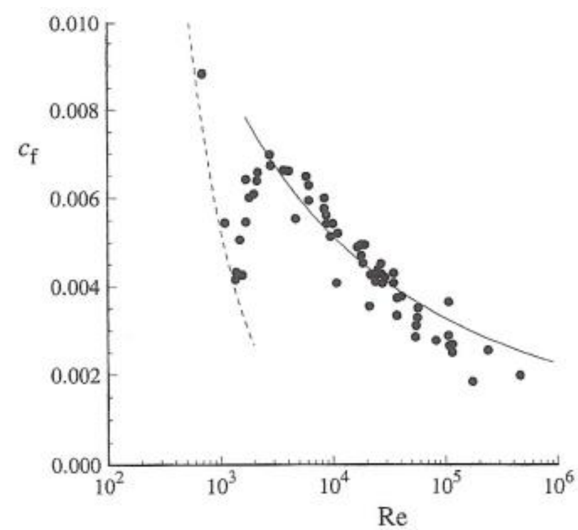


Fig. 7.10. The skin-friction coefficient $c_f \equiv \tau_w/(\frac{1}{2}\rho U_0^2)$ against the Reynolds number ($\mathrm{Re} = 2\bar{U}\delta/\nu$) for channel flow: symbols, experimental data compiled by Dean (1978); solid line, from Eq. (7.55); dashed line, laminar friction law, $c_f = 16/(3\mathrm{Re})$.

Eq. (7) good fit data $Re > 3000$. For $Re < 3000$ log law with universal constants not valid (Patel and Head, 1969).

Re_τ increases almost linearly with Re:

$$Re_\tau \sim 0.09 Re^{0.88}$$

In contrast, velocity ratios increase very slowly with Re:

$$\frac{U_0}{u_\tau} \sim 5 \log_{10} Re$$

Therefore, large fraction increase mean velocity between wall and centerline occurs in viscous wall region.

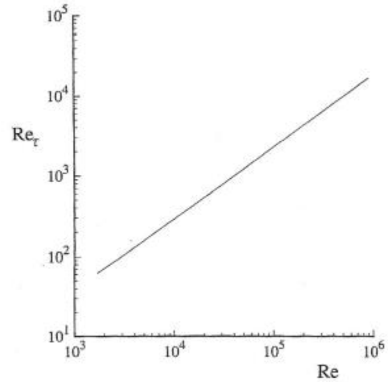


Fig. 7.11. The outer-to-inner lengthscale ratio $\delta/\delta_\nu = Re_\tau$ for turbulent channel flow as a function of the Reynolds number (obtained from Eq. (7.55)).

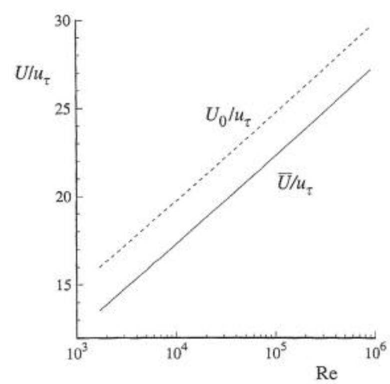


Fig. 7.12. Outer-to-inner velocity-scale ratios for turbulent channel flow as functions of the Reynolds number (obtained from Eq. (7.55)): solid line, $\bar{U}/u_\tau$; dashed line U_0/u_τ.

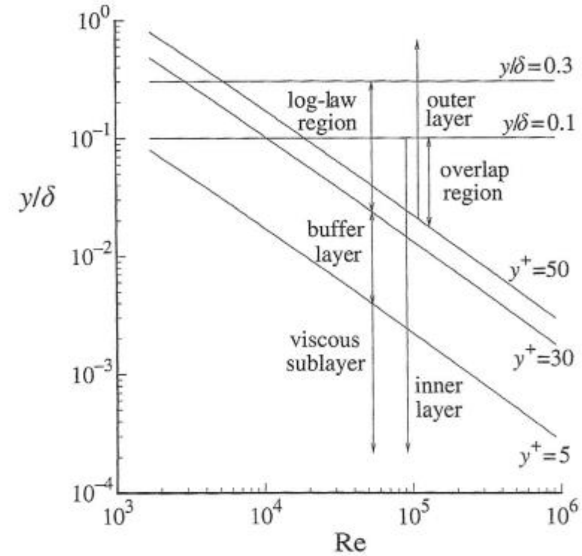


Fig. 7.13. Regions and layers in turbulent channel flow as functions of the Reynolds number.

inner layers y/δ regions vs. Re

Reynolds stresses

Useful to divide flow in three regions:

1) Viscous wall region: $y^+ < 50$
2) Log law region: $50 < y^+ < 120$ $(50\delta_\nu < y < 0.3\delta)$
3) Core region: $y > 0.3\delta$

In 2): self-similarity → $\langle u_i u_j \rangle / k$, production to dissipation ratio P/ε, and normalized mean shear rate Sk/ε all nearly constant, as per Table 7.2

$\langle u_i u_j \rangle / k$ values close to homogeneous shear flow results.

$P/\varepsilon \sim 1$, i.e., viscous, and turbulent transport small.

Table 7.2. *Statistics in turbulent channel flow, obtained from the DNS data of Kim et al. (1987), Re = 13,750*

	Location		
	Peak production $y^+ = 11.8$	Log law $y^+ = 98$	Centerline $y^+ = 395$
$\langle u^2 \rangle / k$	1.70	1.02	0.84
$\langle v^2 \rangle / k$	0.04	0.39	0.57
$\langle w^2 \rangle / k$	0.26	0.59	0.59
$\langle uv \rangle / k$	−0.116	−0.285	0
ρ_{uv}	−0.44	−0.45	0
$\mathcal{S}k/\varepsilon$	15.6	3.2	0
$\mathcal{P}/\varepsilon$	1.81	0.91	0

In 3) mean velocity gradient and shear stress vanish $\rightarrow \frac{Sk}{\varepsilon}, \langle uv \rangle, P \sim 0$.

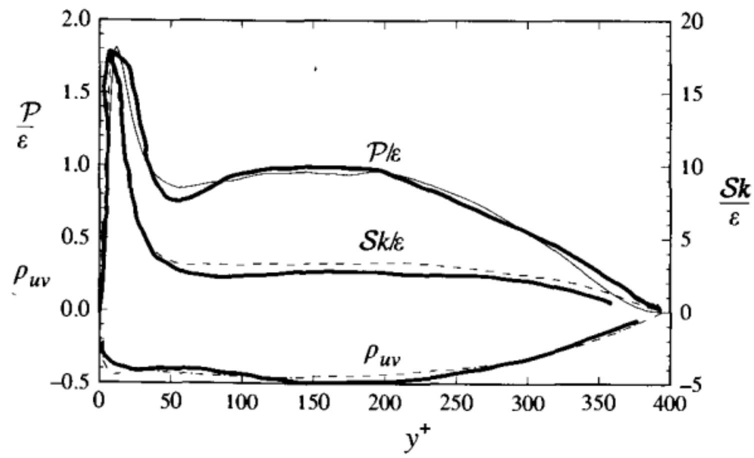


Fig. 7.16. Profiles of the ratio of production to dissipation ($\mathcal{P}/\varepsilon$), normalized mean shear rate ($\mathcal{S}k/\varepsilon$), and shear stress correlation coefficient (ρ_{uv}) from DNS of channel flow at Re = 13,750 (Kim *et al.* 1987).

RS anisotropic but less than in the log law region.

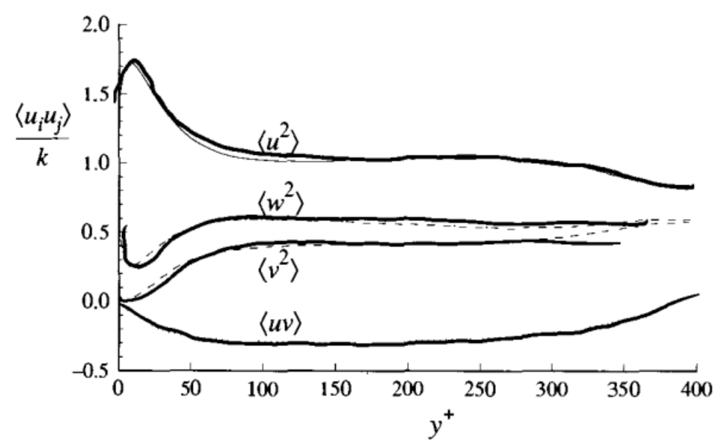


Fig. 7.15. Profiles of Reynolds stresses normalized by the turbulent kinetic energy from DNS of channel flow at Re = 13,750 (Kim *et al.* 1987).

In 1) strongest turbulence: P, ε, k and anisotropy are maximum, with peak values for $y^+ < 20$.

BC $\underline{U}(0) = 0$ determines the behavior of RS for small y (power series):

$$u = a_1 + b_1 y + c_1 y^2 + \cdots$$
$$v = a_2 + b_2 y + c_2 y^2 + \cdots$$
$$w = a_3 + b_3 y + c_3 y^2 + \cdots$$

The coefficients are zero mean random variables and, for fully developed channel flow, are statistically independent of x, z, and t.

For $y = 0$, no-slip condition yields $u = a_1 = 0$ and $w = a_3 = 0$. Similarly, the impermeability condition gives $v = a_2 = 0$.
At the wall, u and w are zero for all x and $z \rightarrow u_x|_{y=0} = w_z|_{y=0}$. Therefore, continuity equation becomes:

$$v_y|_{y=0} = b_2 = 0$$

The significance of b_2 being zero is that close to the wall, there is two-component flow. RS can be obtained by taking products of the power series:

$$\langle u^2 \rangle = \langle b_1^2 \rangle y^2 + \cdots$$
$$\langle v^2 \rangle = \langle c_2^2 \rangle y^4 + \cdots$$
$$\langle w^2 \rangle = \langle b_3^2 \rangle y^2 + \cdots$$
$$\langle uv \rangle = \langle b_1 c_2 \rangle y^3 + \cdots$$

Therefore, $\langle u^2 \rangle, \langle w^2 \rangle$, and k increase from zero as y^2, while $-\langle uv \rangle$ and $\langle v^2 \rangle$ increase more slowly, as y^3 and y^4, respectively.

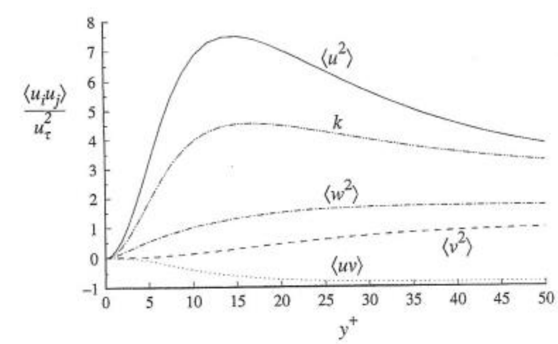


Fig. 7.17. Profiles of Reynolds stresses and kinetic energy normalized by the friction velocity in the viscous wall region of turbulent channel flow: DNS data of Kim *et al.* (1987). Re = 13,750.

TKE equation

$$0 = \underbrace{P}_{\boxed{1}} - \underbrace{\tilde{\varepsilon}}_{\boxed{2}} + \underbrace{\nu \frac{d^2 k}{dy^2}}_{\boxed{3}} - \underbrace{\frac{d}{dy}\langle \frac{1}{2} v \underline{u} \cdot \underline{u} \rangle}_{\boxed{4}} - \underbrace{\frac{1}{\rho}\frac{d}{dy}\langle vp \rangle}_{\boxed{5}}$$

1) Production
2) Pseudo dissipation
3) Viscous diffusion
4) Turbulent convection
5) Pressure transport

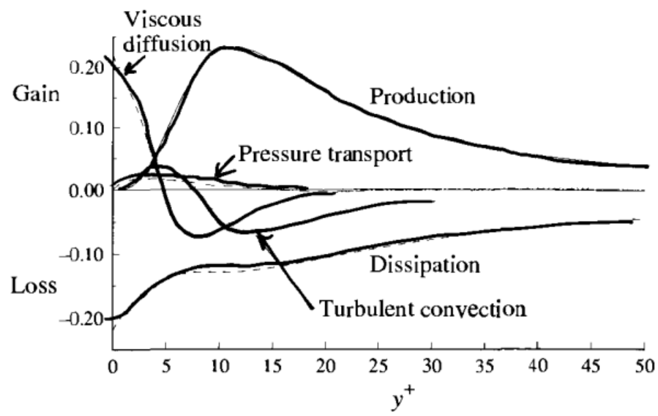


Fig. 7.18. The turbulent-kinetic-energy budget in the viscous wall region of channel flow: terms in Eq. (7.64) normalized by viscous scales. From the DNS data of Kim *et al.* (1987). Re = 13,750.

1) P: y^3 near wall, peak at $y^+ \sim 12$ (occurs where $\mu \langle U \rangle_y = \rho \langle uv \rangle$) and where $P/\varepsilon \sim 1.8$ and excess energy transported away.

5) Small, whereas 4) transport excess P both towards the wall and towards the log law region.

3) Transports towards the wall

2) Is max at wall, where $k = 0$ and $\varepsilon = \tilde{\varepsilon} = \nu k_{yy}|_{y=0}$

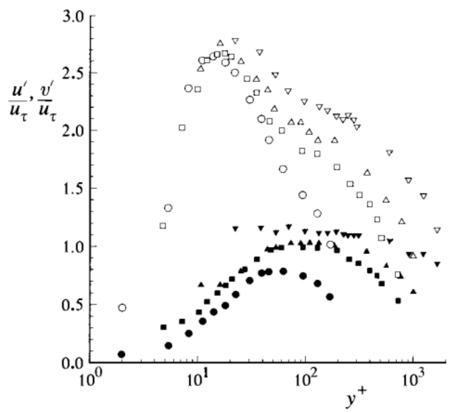


Fig. 7.19. Profiles of r.m.s. velocity measured in channel flow at various Reynolds numbers by Wei and Willmarth (1989). Open symbols: $u'/u_\tau = \langle u^2 \rangle^{1/2}/u_\tau$; ○, $Re_0 = 2{,}970$; □, $Re_0 = 14{,}914$; △, $Re_0 = 22{,}776$; ▽, $Re_0 = 39{,}582$. Solid symbols: $v'/u_\tau = \langle v^2 \rangle^{1/2}/u_\tau$ at the same Reynolds numbers.

Weak $f(Re)$ for $y/\delta < 0.1$, i.e., inner layer

$$rms = \sqrt{\langle u^2 \rangle} = u'$$

u' peak $\neq f(Re)$, but at $y^+ = 50$ there is an increase in u' at higher Re.

Appendix: Channel flow laminar solution

$$U_x + V_y + W_z = 0$$

$$W = 0, U_x = 0, V_y|_0 = 0 \therefore V = 0$$

Momentum equations:

$$0 = -\frac{1}{\rho} p_x + \nu U_{yy}$$

$$0 = -\frac{1}{\rho} p_y \rightarrow p = p(x)$$

$$\frac{dp}{dx} = \frac{dp_w}{dx} = f(x)$$

i.e.,

$0 = -\frac{1}{\rho} p_{w_x} + \nu U_{yy}$ or $\frac{\partial}{\partial y}(\tau) = p_{w_x}$ with $\tau = \mu \frac{\partial}{\partial y}(U)$ = f(y)

$$\frac{\partial}{\partial y}(U_y) = \frac{1}{\mu} p_{w_x}$$

Integrating twice:

$$U_y = \frac{1}{\mu} p_{w_x} y + C_1$$

$$U = \frac{1}{2\mu} p_{w_x} y^2 + C_1 y + C_2$$

Apply BCs:

$$U(0) = 0 \rightarrow C_2 = 0$$

$$U(2\delta) = 0 \rightarrow C_1 = -\frac{1}{\mu} p_{w_x} \delta$$

$$U(y) = \frac{1}{2\mu} p_{w_x} y^2 - \frac{1}{\mu} p_{w_x} \delta y$$

Shear stress:

$$\tau_w = \mu U_y|_0 = -p_{w_x}\delta$$

$$p_{w_x} = -\frac{\tau_w}{\delta}$$

Substituting in the velocity profile:

$$U(y) = -\frac{\tau_w}{\delta}\frac{1}{2\mu}y^2 + \frac{\tau_w}{\delta}\frac{1}{\mu}\delta y$$

$$= \frac{\tau_w}{\mu}\left(y - \frac{y^2}{2\delta}\right)$$

$$= \frac{\tau_w y}{2\mu}\left(2 - \frac{y}{\delta}\right)$$

$$= \frac{\tau_w \delta}{2\rho\nu}\frac{y}{\delta}\left(2 - \frac{y}{\delta}\right)$$

Centerline velocity:

$$U(\delta) = U_0 = \frac{\tau_w\delta}{2\rho\nu} = \frac{\tau_w\delta}{2\mu}$$

Bulk velocity:

$$\overline{U} = \frac{1}{\delta}\int_0^\delta \langle U\rangle dy = \frac{1}{\delta}\int_0^\delta \frac{\tau_w y\delta}{2\delta\rho\nu}\left(2 - \frac{y}{\delta}\right) dy$$

$$= \frac{1}{\delta}\int_0^\delta \frac{\tau_w}{\rho\nu}\left(y - \frac{y^2}{2\delta}\right) dy = \frac{1}{\delta}\frac{\tau_w}{2\rho\nu}\left(\frac{\delta^2}{2} - \frac{\delta^2}{6}\right)$$

$$\overline{U} = \frac{\tau_w\delta}{3\mu}$$

Relation between centerline velocity and bulk velocity:

$$\overline{U} = \frac{2}{3}U_0$$

Relation between Reynolds numbers:

$$Re = \frac{\overline{U}(2\delta)}{\nu} \qquad Re_0 = \frac{U_0 \delta}{\nu}$$

$$Re = \frac{2}{3} U_0 \frac{(2\delta)}{\nu} = \frac{4}{3} Re_0$$

Skin friction coefficients:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} \qquad C_f = \frac{\tau_w}{\frac{1}{2}\rho \overline{U}^2}$$

$$c_f = \frac{2\mu U_0/\delta}{\frac{1}{2}\rho U_0^2} = \frac{4\mu}{\rho U_0 \delta} = \frac{4}{Re_0} = \frac{16}{3Re}$$

$$C_f = \frac{2\mu U_0/\delta}{\frac{1}{2}\rho \overline{U}^2} = \frac{2\mu U_0/\delta}{\frac{1}{2}\rho \frac{4}{9} U_0^2} = \frac{9}{Re_0} = \frac{12}{Re}$$

Friction velocity:

$$u_\tau = \sqrt{\tau_w/\rho}$$

$$\frac{1}{2}\rho U_0^2 c_f = \tau_w \rightarrow \frac{U_0^2 c_f}{2} = \frac{\tau_w}{\rho} = u_\tau^2$$

$$c_f = 2\left(\frac{u_\tau}{U_0}\right)^2 \rightarrow \frac{u_\tau}{U_0} = \sqrt{\frac{c_f}{2}} = \sqrt{\frac{2}{Re_0}} = \sqrt{\frac{8}{3Re}}$$

$$Re_{max} = 1350 \rightarrow \frac{u_\tau}{U_0} = \sqrt{\frac{8}{3 \times 1350}} = 0.044$$