Chapter 8: Channel and Pipe Flow (Chap. 7 Bernard)

Channel, pipe, and BL similar flows due wall boundary, especially near wall, however some differences due differences in their outer flows.

Pipe curvature effects not discernable.

Channel flow experiments difficult due requirement large span with 2D mean flow vs. DNS which can use periodic boundary conditions. Whereas pipe flow amendable both.

BL amendable both experiments and DNS and better for experimental study of coherent structures and transition to fully turbulent flow.

Pat 1: Channel flow

Flow between two parallel plates, with constant P_x: Poiseuille flow.

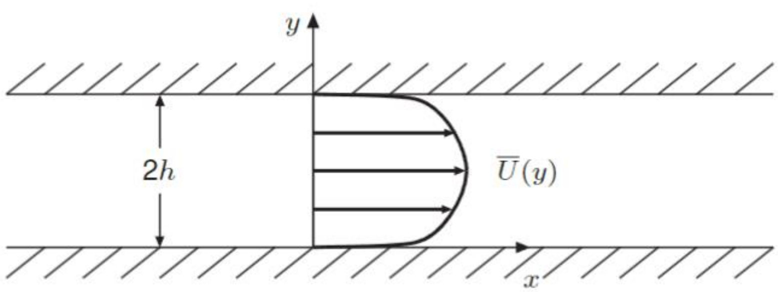


Figure 7.1 Geometry of channel flow.

For fully developed laminar flow:

$$U(y) = -\frac{1}{2\mu}\frac{\partial P}{\partial x}y(2h - y)$$

This solution holds for $R_e = hU_m/\nu < 1000$, where:

$$U_m = \frac{1}{2h}\int_0^{2h}\overline{U}(y)dy = \text{mean bulk velocity}$$

For turbulent flow $\underline{\overline{U}} = \left(\overline{U}(y), 0,0\right)$ and $\underline{u} = (u, v, w)$.

Periodic BCs in x, z assuming large enough domain such that, e.g., $f(r) \to 0$ for large r.

Channel flow simulations characterized using:

$$R_\tau = \frac{U_\tau h}{\nu} = \frac{h}{(\nu/U_\tau)}$$

$h =$ length scale channel $\frac{\nu}{U_\tau} =$ viscous length scale = size flow features near wall viscous region

Based on the friction velocity

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

Where:

$$\tau_w = \mu \frac{d\overline{U}}{dy}(0)$$

Is the wall shear stress.

For large R_τ clear separation inner and outer flow.

R_e	3300	125000
R_τ	180	5186
DNS (year)	1987	2015

Reynolds Stress and Force Balance

For fully developed mean flow, momentum equations become:

$$0 = -\frac{\partial \overline{P}}{\partial x} + \frac{d}{dy}\left(\mu \frac{d\overline{U}}{dy} - \rho\overline{uv}\right) = -\frac{\partial \overline{P}}{\partial x} + \frac{d\tau}{dy} \quad (1) \quad \boxed{x\text{ −direction}}$$

$$\tau = \mu \frac{d\overline{U}}{dy} - \rho\overline{uv} = \text{total mean shear stress}$$

$$0 = -\frac{\partial \overline{P}}{\partial y} - \rho \frac{d\overline{v^2}}{dy} \quad (2) \quad \boxed{y\text{ −direction}}$$

$$0 = 0 \quad \boxed{z\text{ −direction}}$$

Note that $\overline{U}, \overline{u^2}, \overline{v^2}, \overline{uv} = f(y)$.

Taking an x derivative of Eqs. (1) and (2) shows that

$$\frac{\partial \overline{P}}{\partial x} \neq f(x,y) = \text{constant} \ \text{(i.e., } \overline{P}_{xx} = \overline{P}_{yx} = \overline{P}_{xy} = 0\text{)}$$

Integration of Eq. (2) across the channel from 0 to y:

$$\overline{P}(x,y) = \overline{P}(x,0) - \rho\overline{v^2}(y)$$

Since $\overline{v^2}(0) = 0$, showing that $\overline{P}(x,y)$ is minimum where $\overline{v^2}(y)$ is maximum, which differs from laminar flow where the pressure is constant across the flow.

Also, since $\frac{\partial \overline{P}}{\partial x} = \text{constant}$,

$$\overline{P}(x,y) - \overline{P}(x+L,y) \neq f(y)$$

Integration of Eq. (1) over the area $0 \leq x \leq L, 0 \leq y \leq 2h$ yields force balance:

$$\int_0^L \int_0^{2h} 0\, dxdy = \int_0^L \int_0^{2h} \left[-\frac{\partial \overline{P}}{\partial x} + \frac{d}{dy}\left(\mu \frac{d\overline{U}}{dy} - \rho \overline{uv} \right) \right] dxdy$$

$$\Delta\overline{P} 2h - \tau_w 2L = 0 \quad (3)$$

Where:

$$\Delta\overline{P} = -L \frac{\partial \overline{P}}{\partial x} = \overline{P}(x,0) - \overline{P}(x+L,0)$$

Is the pressure drop between x locations. Note that in deriving Eq. (3) the channel centerline asymmetry condition was used:

$$\frac{d\overline{U}}{dy}(0) = -\frac{d\overline{U}}{dy}(2h)$$

Eq. (3) shows that pressure force is balanced by τ_w force. For turbulent flow, channel center high momentum fluid is better able to penetrate wall region vs. laminar flow resulting in steeper velocity gradient near the wall.

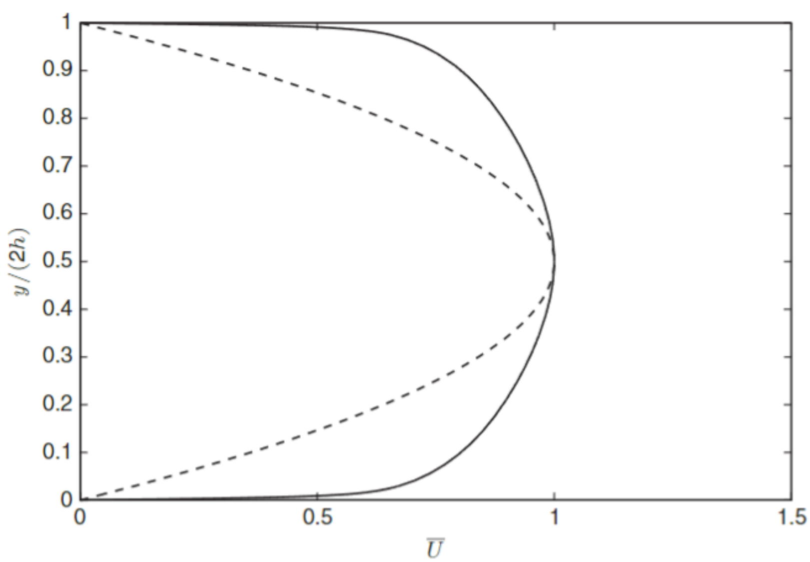


Figure 7.2 Average velocity in channel flow of width $2h$ scaled by mean centerline velocity, U_{cl}: —, turbulent flow; – –, laminar flow.

Differentiating the velocity profile of fully developed laminar flow $[U(y) = -\frac{1}{2\mu}\frac{\partial P}{\partial x}y(2h-y)]$ and substituting $\tau_w = \mu\frac{dU}{dy}(0)$ gives:

$$\frac{dU}{dy} = -\frac{1}{2\mu}\frac{\partial P}{\partial x}(2h-2y) = -\frac{1}{\mu}\frac{\partial P}{\partial x}(h-y)$$

$$= -\frac{1}{\mu}\frac{\partial P}{\partial x}h\left(1-\frac{y}{h}\right)$$

$$\mu U_{yy} = \tau_y = P_x$$

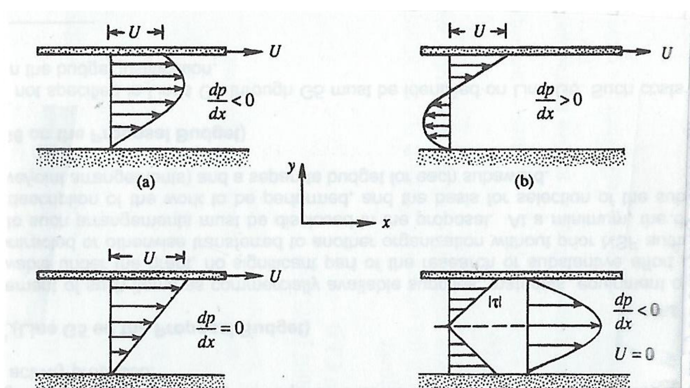
$$\tau = \mu U_y$$

$$\frac{dU}{dy}(0) = -\frac{1}{2\mu}\frac{\partial P}{\partial x}2h = \frac{\tau_w}{\mu} \rightarrow -\frac{\partial P}{\partial x} = \frac{\tau_w}{h}$$

$$\tau_w = -h\frac{\partial P}{\partial x}$$

$$\mu\frac{dU}{dy} = -h\frac{\partial P}{\partial x}\left(1-\frac{y}{h}\right) = \tau_w\left(1-\frac{y}{h}\right)$$

i.e., the shear stress $\tau_{12}=\mu\frac{dU}{dy}$ is linear across the channel: momentum flux (shear stress) across channel from the centerline towards walls due to $-P_x$.



(c) Plane Couette flow
(d) Plane Poiseuille flow

Figure 9.4 Various cases of parallel flow in a channel.

For turbulent flow:

$$-L\frac{\partial \overline{P}}{\partial x}2h - 2\tau_w L = 0$$

$$-\frac{\partial \overline{P}}{\partial x} = \frac{\tau_w}{h} = \text{constant} \quad (4)$$

Substituting Eq. (4) into Eq. (1) and integrating from 0 to y, gives:

$$0 = \int_0^y \left[\frac{\tau_w}{h} + \frac{d}{dy}\left(\mu\frac{d\overline{U}}{dy} - \rho\overline{uv}\right)\right] dy$$

$$0 = \frac{\tau_w}{h}y + \left[\mu\frac{d\overline{U}}{dy} - \rho\overline{uv}\right]_0^y$$

$$0 = \frac{\tau_w}{h}y - \tau_w + \mu\frac{d\overline{U}}{dy} - \rho\overline{uv}$$

$$\mu\frac{d\overline{U}}{dy} - \rho\overline{uv} = \tau_w\left(1 - \frac{y}{h}\right) \quad (5)$$

i.e., same as laminar flow, with the addition of $-\rho\overline{uv}$. i.e., sum of viscous and turbulent stress varies linearly across the channel.

Eq. (5) can be scaled using the friction velocity, such that:

$$\nu \frac{d\overline{U}}{dy} - \overline{uv} = U_\tau^2 \left(1 - \frac{y}{h}\right)$$

$$\nu \frac{d\overline{U}^+}{dy} - \frac{\overline{uv}}{U_\tau} = U_\tau \left(1 - \frac{y}{h}\right)$$

$$\nu \frac{d\overline{U}^+}{dy^+} \frac{U_\tau}{\nu} - \frac{\overline{uv}}{U_\tau} = U_\tau \left(1 - \frac{y}{h}\right)$$

$$\frac{d\overline{U}^+}{dy^+} - \overline{uv}^+ = 1 - \frac{y}{h} = 1 - \frac{y^+ \nu U_\tau}{U_\tau \nu R_\tau}$$

$$R_\tau = \frac{U_\tau h}{\nu}$$

$\frac{d\overline{U}^+}{dy^+} - \overline{uv}^+ = 1 - \frac{y^+}{R_\tau} = \overline{\tau}^+$ = total scaled mean shear stress (of course also linear)

Where:

$$\overline{U}^+ = \frac{\overline{U}}{U_\tau} \qquad \overline{uv}^+ = \frac{\overline{uv}}{U_\tau^2} \qquad y^+ = \frac{U_\tau y}{\nu}$$

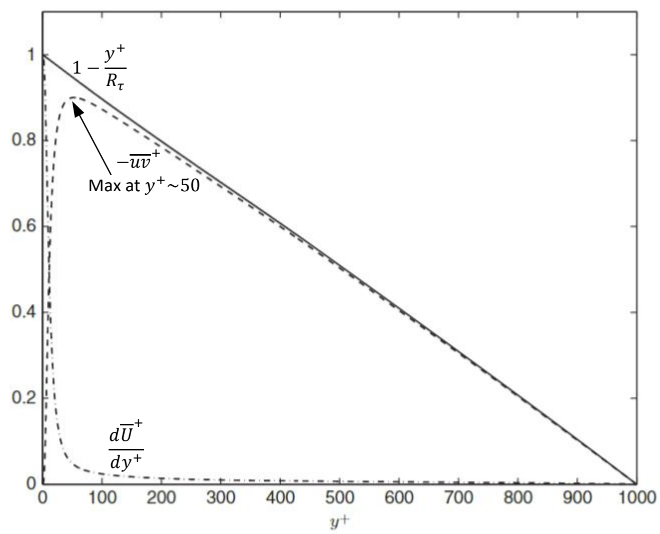


Figure 7.3 Decomposition of the total stress as given by Eq. (7.14) in turbulent channel flow: – · –, $d\overline{U}^+/dy^+$; – –, $-\overline{uv}^+$; —, $1 - y^+/R_\tau$. Data taken from [13].

The mean viscous (molecular) momentum transport/flux is confined to a thin layer near the wall. The drop in molecular momentum transport/flux is compensated by the turbulent momentum transport/flux, which is asymmetric across the channel. The peak in $-\overline{uv}$ is at $y^+ \approx 53$ after which it has a nearly linear variation to zero at the channel centerline where the mean shear is zero.

Eq. (1) scaling:

$$0 = \frac{\tau_w}{h} + \frac{d}{dy}\left(\mu\frac{d\overline{U}}{dy} - \rho\overline{uv}\right)$$

$$0 = \frac{\rho U_\tau^2}{h} + \frac{d}{dy}\left(\mu\frac{d\overline{U}}{dy} - \rho\overline{uv}\right)$$

$$0 = \frac{U_\tau^2}{h} + \frac{d}{dy^+}\frac{U_\tau}{\nu}\left(\nu\frac{d\overline{U}}{dy^+}\frac{U_\tau}{\nu} - \overline{uv}\right)$$

$$0 = \frac{\nu U_\tau}{h} + \frac{d}{dy^+}\left(U_\tau\frac{d\overline{U}}{dy^+} - \overline{uv}\right)$$

$$0 = \frac{\nu U_\tau}{h} + U_\tau\frac{d}{dy^+}\left(\frac{d\overline{U}}{dy^+} - \frac{\overline{uv}}{U_\tau}\right)$$

$$0 = \frac{\nu}{hU_\tau} + \frac{d}{dy^+}\left(\frac{1}{U_\tau}\frac{d\overline{U}}{dy^+} - \frac{\overline{uv}}{U_\tau^2}\right)$$

$$0 = \frac{1}{R_\tau} + \frac{d}{dy^+}\left(\frac{d\overline{U}^+}{dy^+} - \overline{uv}^+\right)$$

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$y^+ = \frac{U_\tau y}{\nu}$$

$$\frac{d}{dy} = \frac{d}{dy^+}\frac{U_\tau}{\nu}$$

$$R_\tau = \frac{U_\tau h}{\nu}$$

$$\overline{U}^+ = \frac{\overline{U}}{U_\tau}$$

0 = (1) pressure force + (2) viscous force – (3) turbulence force

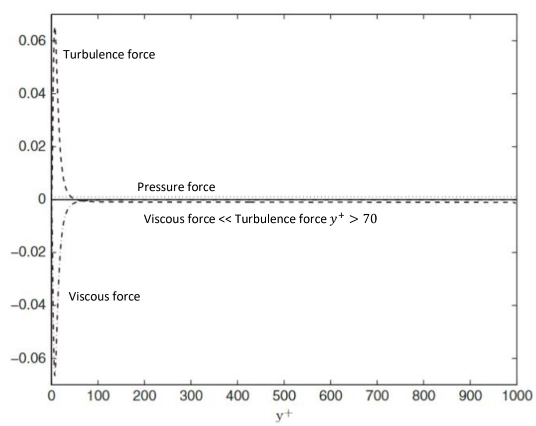


Figure 7.4 Decomposition of the mean momentum equation (7.16) in turbulent channel flow: – · –, viscous force; ––, turbulent transport; · · ·, pressure force. Data taken from [13].

Pressure force (1) Is constant across channel and balanced by (3) for $y^+ \geq$ approximately 70.

Near wall $y^+ \leq$ approximately 70, complex physics wherein (3) transports momentum from outer channel towards wall (gain), which is counterbalanced by viscous diffusion (2) again towards the wall (loss).

Mean Flow Similarity: flow field regions.

1) Viscous sublayer: $f(\mu)$
2) Channel center = outer/core region $\neq\ f(\mu)$
3) Overlap layer = intermediate region, requires high Re for separation of 1) and 2).

Between 1) and 3) = buffer layer where turbulence is maximum: $5 \leq y^{+} \leq 30$, as per later discussion.

Viscous sublayer (see Appendix for RS for u_i and $\langle u_i u_j \rangle$)

Viscosity essential for flow near solid boundaries. Evaluating Eq. (1) at $y = 0$ and using Eq. (4) gives:

$$0 = \frac{\tau_w}{h} + \mu \frac{d^2\overline{U}}{dy^2}(0) - \rho \cancel{\frac{d\overline{uv}(0)}{dy}} = \frac{\mu}{h}\frac{d\overline{U}}{dy}(0) + \mu \frac{d^2\overline{U}}{dy^2}$$

$$\frac{d^2\overline{U}}{dy^2}(0) = -\frac{1}{h}\frac{d\overline{U}}{dy}(0) \quad (6)$$

Where the fact that

$$\frac{d\overline{uv}(0)}{dy} = 0$$

Follows from the identity:

$$\frac{\partial \overline{uv}}{\partial y} = \overline{\frac{\partial u}{\partial y} v} + \overline{u \frac{\partial v}{\partial y}}$$

Differentiating Eq. (1) with respect to y:

$$\frac{d^3\overline{U}}{dy^3}(0) = 0 \quad (7)$$

Since

$$\frac{\partial^2 \overline{uv}}{\partial y^2}(0) = 0$$

$$\frac{\partial^2 \overline{uv}}{\partial y^2} = \frac{\partial}{\partial y}\left(\overline{\frac{\partial u}{\partial y} v} + \overline{u \frac{\partial v}{\partial y}}\right) = 2\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}} + \overline{u \frac{\partial^2 v}{\partial y^2}} + \overline{\frac{\partial^2 u}{\partial y^2} v}$$

$$= -2\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}} + \overline{u \frac{\partial^2 v}{\partial y^2}} + \overline{\frac{\partial^2 u}{\partial y^2} v} = 0 \ at \ y = 0$$

Using continuity $v_y = -u_x$ and $w_z = 0$.

Taylor series expansion for $\overline{U}(y)$ near y = 0:

$$\overline{U}(y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} \frac{d^n \overline{U}}{dy^n}(0) \quad (8)$$

Substituting Eqs. (6) and (7) into (8) gives

$$\overline{U}(y) = h \frac{d\overline{U}}{dy}(0) \frac{y}{h} + h^2 \ \frac{d^2 \overline{U}}{dy^2}(0) \frac{y^2}{2h^2} + O\left(\frac{y}{h}\right)^4$$

$$\overline{U}(y) \approx h \frac{d\overline{U}}{dy}(0) \frac{y}{h} - h \frac{d\overline{U}}{dy}(0) \ \frac{y^2}{2h^2}$$

$$\overline{U}(y) \approx h \frac{d\overline{U}}{dy}(0) \left(\frac{y}{h} - \frac{y^2}{2h^2}\right) \quad (9)$$

Scaling Eq. (9) with U_τ, yields

$$\overline{U}^+(y^+) = y^+ - \frac{(y^+)^2}{2R_\tau} + \cdots$$

For small $y^+ \to \overline{U}^+ = y^+$, EFD and DNS $y^+ < 5$.

Using dimensional analysis:

$$\overline{U} = f(y, \tau_w, \rho, \nu) \to \overline{U}^+ = f(y^+)$$

Represents the law of the wall.

$$\overline{U}^+ = y^+ \neq f(R_e) \ \therefore \text{Complete similarity}$$

Intermediate layer

Near $y^+ \approx 50$, $\frac{d\overline{U}^+}{dy^+}$ minimum and $\overline{uv}^+$can be assumed nearly constant in this region, as per Fig. 7.3 and $\frac{d\overline{U}^+}{dy^+} - \overline{uv}^+ = 1 - \frac{y^+}{R_\tau}$, such that:

$$\overline{uv}^+ \approx -1$$

Or equivalently

$$\overline{uv} \approx \frac{\tau_w}{\rho} \quad (9)$$

If Eq. (9) is legitimate, then dimensional analysis suggests that:

$$\frac{d\overline{U}}{dy} = f(y, \tau_w, \rho) \ \propto \frac{U_\tau}{y} \quad (10)$$

Introducing a dimensionless constant of proportionality k, known as the Von Karman constant, Eq. (10) becomes

$$\frac{d\overline{U}}{dy} = \frac{U_\tau}{ky} \quad (11)$$

Expressing Eq. (11) in wall units and integrating gives

$$\frac{1}{U_\tau}\frac{d\overline{U}}{dy} = \frac{1}{ky}$$

$$\frac{d\overline{U}^+}{dy^+}\frac{U_\tau}{\nu} = \frac{U_\tau}{ky^+\nu}$$

$$\frac{d\overline{U}^+}{dy^+} = \frac{1}{ky^+}$$

$$\int \frac{d\overline{U}^+}{dy^+}dy^+ = \int \frac{1}{ky^+}dy^+$$

$$\overline{U}^+(y^+) = \frac{1}{k}\log y^+ + B \quad (12)$$

$k = 0.41$

$B = 5.2$

Where B is a constant.

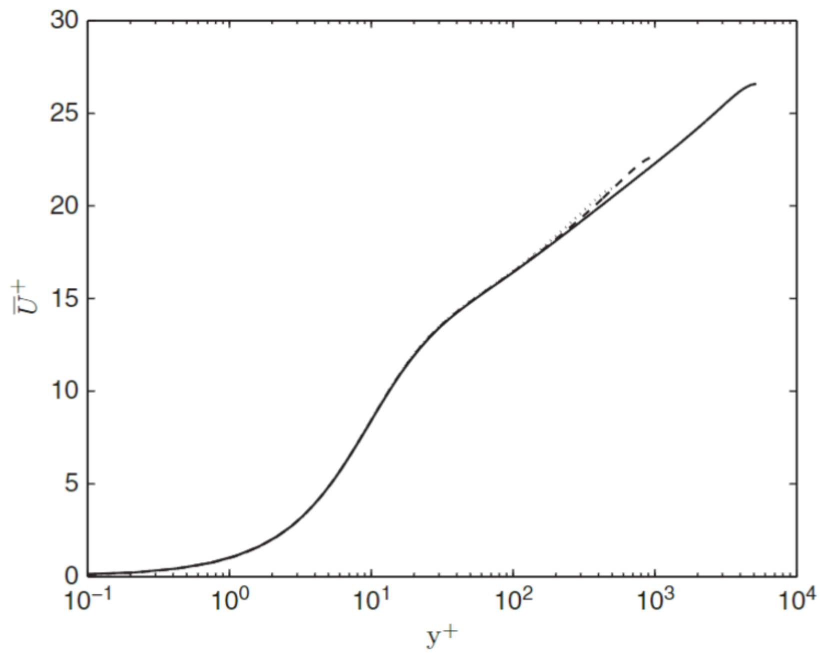
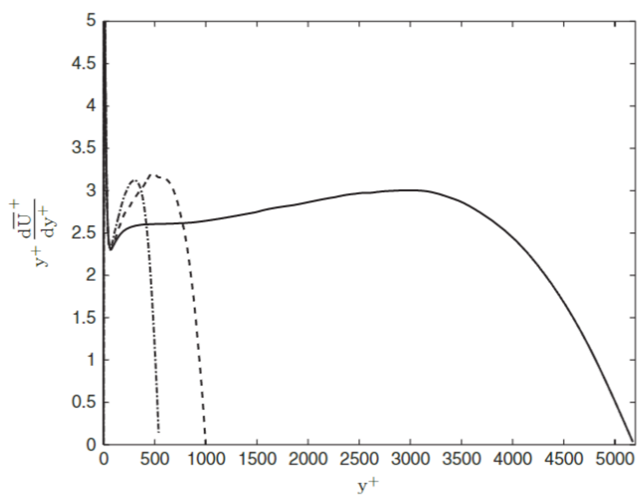


Figure 7.5 Semi-log plot of $\overline{U}^+$ showing an approximate log-law behavior. · · ·, $R_\tau = 541$; – –, $R_\tau = 1000$; —, $R_\tau = 5186$. Data from [10, 13].

More precise determination of whether a log law is present can be obtained by examination of

$$\beta = y^+ \frac{d\overline{U}^+}{dy^+} = \frac{1}{k}$$

Which will be constant, and equal to $1/k$, in log-law regions if they exist.



1/0.41=2.44

Figure 7.6 β as defined in Eq. (7.32) for the mean velocities in Figure 7.5. – · –, $R_\tau = 541$; – –, $R_\tau = 1000$; —, $R_\tau = 5186$. Data from [10, 13].

β constant only for $R_\tau = 5186$ → intermediate layer developed only for $R_\tau > 2200$.

Buffer layer: Merges smoothly the viscosity-dominated sub-layer and turbulence-dominated log-layer in the region 5< $y^+ \le 30$.

Unified Inner layer: There are several ways to obtain composite of sub-/buffer and log-layers.

Evaluating the RANS equation near the wall using μ_t turbulence model shows that:

$$\mu_t \sim y^3 \quad y \to 0$$

Several expressions which satisfy this requirement have been derived and are commonly used in turbulent-flow analysis. That is:

$$\mu_t = \mu \kappa e^{-\kappa B}\left[e^{\kappa U^+} - 1 - \kappa U^+ - \frac{(\kappa U^+)^2}{2}\right]$$

Assuming the total shear is constant very near to the wall a composite formula which is valid in the sub-layer, blending layer, and logarithmic-overlap regions is obtained:

$$U^+ = y^+ - e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa U^+ - \frac{(\kappa U^+)^2}{2} - \frac{(\kappa U^+)^3}{6}\right]$$

Fig. 6-11 shows a comparison of this equation with experimental data obtained very close to the wall. The agreement is excellent. It should be recognized that obtaining data this close to the wall is very difficult.

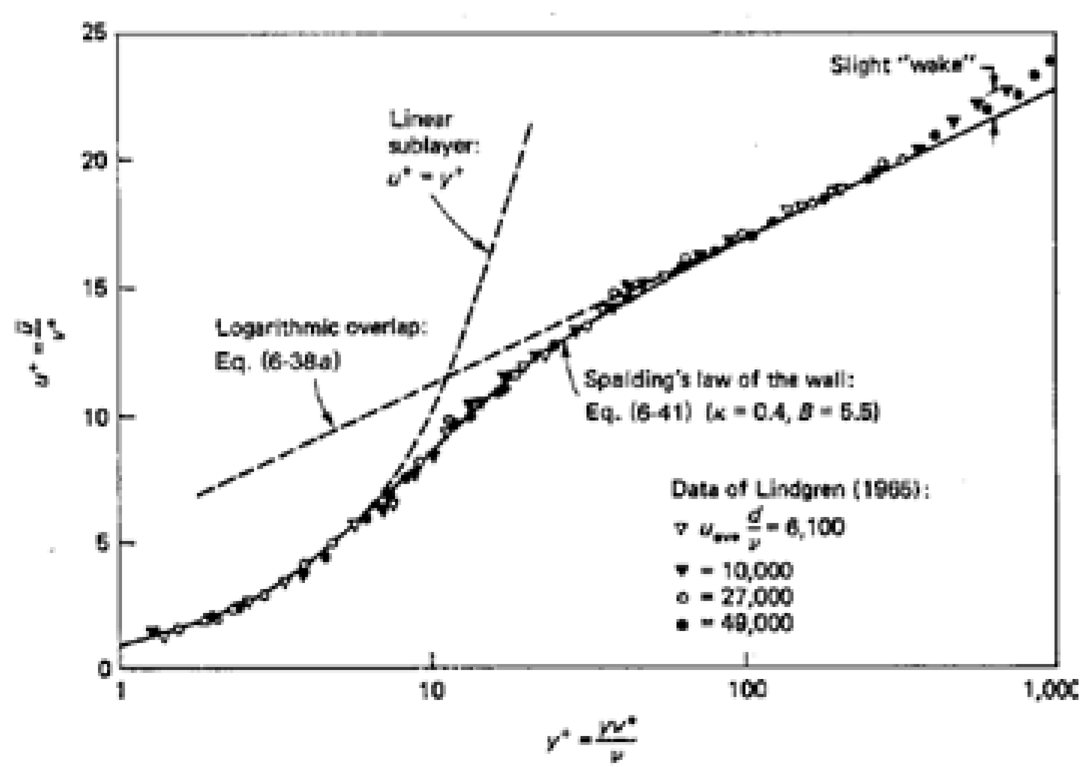


FIGURE 6-11
Comparison of Spalding's inner-law expression with the pipe-flow data of Lindgren (1965).

Velocity moments

Effect of Re negligible in core region of the channel.

Near wall, for larger R_τ, more rapid changes in correlations and shifted closer to the wall.

$\overline{v^2}$ damped near wall.

$\overline{v^2}$ and $\overline{w^2}$ somewhat isotropic in core region, but not $\overline{u^2}$.

$\overline{u^2}_{max}$ @ $y^+ \sim 15.5$

Buffer layer: steep $\overline{U_y}$ and max $\overline{u_i u_j}$

For $y^+ < 2.5$ results seem independent of R_e. However, for $R_\tau = 5186$:

$$\frac{du_{rms}^+}{dy^+}(0) = 0.5$$

$$\frac{d^2 u_{rms}^+}{dy^{+2}}(0) = -0.038$$

$R_\tau = 1000$:

$$\frac{du_{rms}^+}{dy^+}(0) = 0.47$$

$$\frac{d^2 u_{rms}^+}{dy^{+2}}(0) = -0.032$$

i.e., persistent Reynolds number effect.

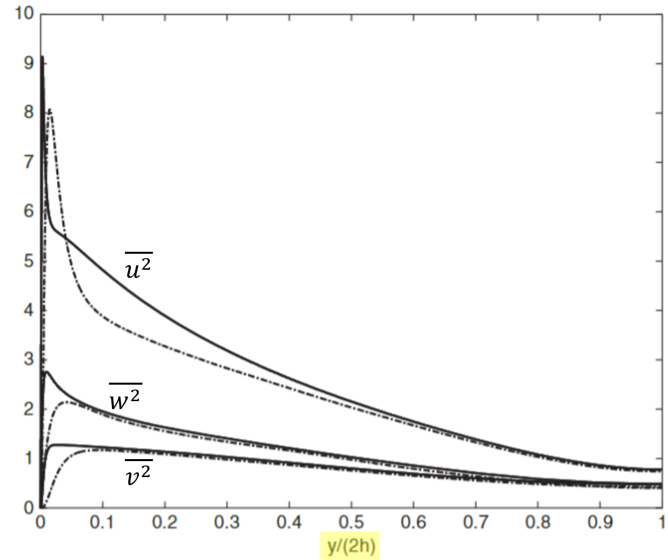


Figure 7.7 Normal Reynolds stresses in channel flow. —, $R_\tau = 5186$; – · –, $R_\tau = 1000$. Top curves are $\overline{u^2}$, middle curves are $\overline{w^2}$, lower curves are $\overline{v^2}$. Data taken from [10, 13].

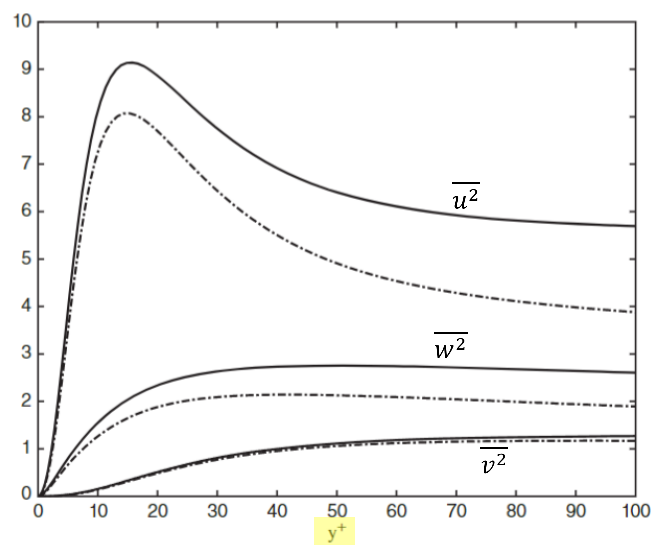


Figure 7.8 Normal Reynolds stresses in channel flow plotted with respect to y^+. —, $R_\tau = 5186$; – · –, $R_\tau = 1000$. Top curves are $\overline{u^2}$, middle curves are $\overline{w^2}$, lower curves are $\overline{v^2}$. Data taken from [10, 13].

$\overline{U}$ is linear up to $y^+ = 5$, so consider Taylor series expansion of u_{rms}^+ about $y = 0$:

$$u_{rms}^+ = \underbrace{\cancel{u_{rms}^+(0)}}_{\text{no-slip}} + \frac{du_{rms}^+}{dy^+}(0)y^+ + \frac{1}{2}\frac{d^2 u_{rms}^+}{dy^{+2}}(0)y^{+2} + O((y^+)^3) \quad (13)$$

The distance from the boundary over which u^+_{rms} can be modeled as linear can be analyzed considering the ratio $u^+_{rms}/\overline{U}^+$.

Recall scaling of $\overline{U}(y)$ resulted in the equation:

$$\overline{U}^+(y^+) = \frac{\overline{U}(y)}{U_\tau} = y^+ - \frac{(y^+)^2}{2R_\tau} + O((y^+)^4)$$

$$\overline{U}^+(y^+) \approx y^+\left(1 - \frac{y^+}{2R_\tau}\right) \quad (14)$$

Dividing Eq. (13) by $\overline{U}^+$ and combining with Eq. (14) gives:

$$\frac{u^+_{rms}}{\overline{U}^+} = \frac{du^+_{rms}}{dy^+}(0)\frac{y^+}{\overline{U}^+} + \frac{1}{2}\frac{d^2u^+_{rms}}{dy^{+2}}(0)\frac{y^{+2}}{\overline{U}^+} + O\left(\frac{(y^+)^3}{\overline{U}^+}\right)$$

$$\frac{u^+_{rms}}{\overline{U}^+} = \frac{du^+_{rms}}{dy^+}(0)\frac{y^+}{y^+\left(1 - \frac{y^+}{2R_\tau}\right)} + \frac{1}{2}\frac{d^2u^+_{rms}}{dy^{+2}}(0)\frac{y^{+2}}{y^+\left(1 - \frac{y^+}{2R_\tau}\right)} + O((y^+)^2)$$

Define $y^{+\prime} = y^+/2R_\tau$

$$\frac{u^+_{rms}}{\overline{U}^+} = \frac{du^+_{rms}}{dy^+}(0)\frac{y^+}{y^+(1 - y^{+\prime})} + \frac{1}{2}\frac{d^2u^+_{rms}}{dy^{+2}}(0)\frac{y^{+2}}{y^+(1 - y^{+\prime})} + O((y^+)^2)$$

and use binomial theorem such that:

$$\frac{1}{(1 - y^{+\prime})} \approx 1 + y^{+\prime}$$

Therefore,

$$\frac{u^+_{rms}}{\overline{U}^+} = \frac{du^+_{rms}}{dy^+}(0)\left(1 + \frac{y^+}{2R_\tau}\right) + \frac{1}{2}\frac{d^2u^+_{rms}}{dy^{+2}}(0)y^+\left(1 + \frac{y^+}{2R_\tau}\right) + O((y^+)^2)$$

$$\frac{u^+_{rms}}{\overline{U}^+} = \frac{du^+_{rms}}{dy^+}(0) + y^+\left(\frac{1}{2R_\tau}\frac{du^+_{rms}}{dy^+}(0) + \frac{1}{2}\frac{d^2u^+_{rms}}{dy^{+2}}(0)\right) + O((y^+)^2)$$

It follows that near the wall:

$$\frac{u_{rms}^+}{\overline{U}^+} = 0.5 + y^+\left(\frac{0.25}{R_\tau} - 0.019\right) + \cdots$$

i.e., linearity maintained until $y^+ \approx 2$.

Similarly, spanwise rms fluctuations are given in the form:

$$w_{rms}^+ = \underbrace{\cancel{w_{rms}^+(0)}}_{\boxed{\text{no-slip}}} + \frac{dw_{rms}^+}{dy^+}(0)y^+ + \frac{1}{2}\frac{d^2w_{rms}^+}{dy^{+2}}(0)y^{+2} + O((y^+)^3)$$

And computations show that at the wall approximate expression is:

$$w_{rms}^+ = 0.25y^+ + \cdots$$

Using continuity, it can be shown that:

$$\frac{dv_{rms}^+}{dy^+} = 0$$

Such that, near the wall

$$v_{rms}^+ = \frac{1}{2}\frac{d^2v_{rms}^+}{dy^{+2}}(0)y^{+2} + O((y^+)^3)$$

And computations show that near the wall approximate expression is:

$$v_{rms}^+ = 0.006y^{+2} + O((y^+)^3)$$

TKE budget

Simplification of TKE equation obtained in Chapter 3 for channel flow yields:

$$0 = \underbrace{-\overline{uv}\frac{d\overline{U}}{dy}}_{\boxed{1}} - \underbrace{\varepsilon}_{\boxed{2}} - \frac{1}{\rho}\overline{pv_y} + \underbrace{\nu\frac{d^2k}{dy^2}}_{\boxed{4}} - \frac{1}{2}\underbrace{\frac{d\overline{vu_j^2}}{dy}}_{\boxed{5}}$$

1) Production
2) Dissipation
3) Pressure work/transport
4) Viscous diffusion/transport
5) Turbulent transport = $\frac{1}{2}\langle v\underline{u}\cdot\underline{u}\rangle_y$

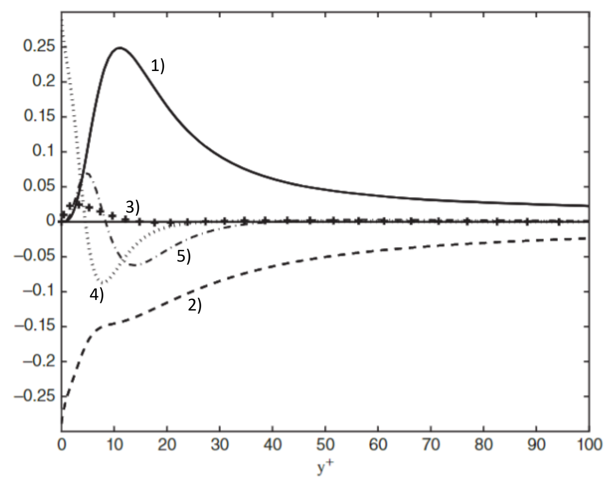


Figure 7.9 Turbulent kinetic energy budget in channel flow $R_\tau = 5186$ [10] scaled with ν and u_τ: —, production; – –, dissipation; +, pressure work; · · ·, viscous diffusion; – · –, turbulent transport.

For $y^+ > 30$ up to $y/2h = 1$, $P \approx \varepsilon$.

P_{max} @ $y^+ = 12$ (near k peak), $P/\varepsilon \sim 1.8$

ε_{max} @ $y^+ = 0$, i.e., at the wall and has local plateau near P_{max}.

Turbulent transport (5) is negative for $8 < y^+ < 30$ and positive for $y^+ < 8 \rightarrow$ large P/ε transported towards wall.

At wall:

$$\varepsilon = \nu\frac{d^2k}{dy^2}$$

i.e., dissipation equals molecular diffusion.

Most complex physics is in buffer layer $5 < y^+ < 30$.

Location of P_{max} can be estimated by rewriting the production term as:

$$-\overline{uv}^+\frac{d\overline{U}^+}{dy^+} = \left(1 - \frac{y^+}{R_\tau} - \frac{d\overline{U}^+}{dy^+}\right)\frac{d\overline{U}^+}{dy^+} \quad (15)$$

P_{max} is located where d/dy^+ of Eq. (15) equals zero:

$$\frac{d}{dy^+}\left[\left(1-\frac{y^+}{R_\tau}-\frac{d\overline{U}^+}{dy^+}\right)\frac{d\overline{U}^+}{dy^+}\right]=0$$

$$\left(-\frac{1}{R_\tau}-\frac{d^2\overline{U}^+}{dy^{+2}}\right)\frac{d\overline{U}^+}{dy^+}+\left(1-\frac{y^+}{R_\tau}-\frac{d\overline{U}^+}{dy^+}\right)\frac{d^2\overline{U}^+}{dy^{+2}}=0$$

$$-\frac{1}{R_\tau}\frac{d\overline{U}^+}{dy^+}+\left(1-\frac{y^+}{R_\tau}-2\frac{d\overline{U}^+}{dy^+}\right)\frac{d^2\overline{U}^+}{dy^{+2}}=0$$

$$\left(1-2\frac{d\overline{U}^+}{dy^+}\right)\frac{d^2\overline{U}^+}{dy^{+2}}=0$$

Where terms $O(R_\tau^{-1})$ are dropped.

$$\frac{d\overline{U}^+}{dy^+}-\overline{uv}^+=1-\frac{y^+}{R_\tau}$$

If $y^+ \ll R_\tau \rightarrow$

$$\frac{d\overline{U}^+}{dy^+}-\overline{uv}^+=1$$

Since $\frac{d^2\overline{U}^+}{dy^{+2}} \neq 0$ in the region of P_{max}, then

$$\frac{d\overline{U}^+}{dy^+}=\frac{1}{2}=-\overline{uv}^+ \quad (16)$$

Point where Eq. (16) is satisfied is visible in Fig. 7.3 at $y^+ \approx 12$.

ε budget

$$\frac{D\varepsilon}{Dt} = P_\varepsilon^1 + P_\varepsilon^2 + P_\varepsilon^3 + P_\varepsilon^4 + \Pi_\varepsilon + T_\varepsilon + D_\varepsilon - \Upsilon_\varepsilon$$

$P_\varepsilon =$ production

$\Upsilon_\varepsilon =$ dissipation of dissipation

$\Pi_\varepsilon + T_\varepsilon + D_\varepsilon =$ redistribution

Like homogeneous shear flow for large y^+(away from wall)

$$P_\varepsilon^4 \sim \Upsilon_\varepsilon$$

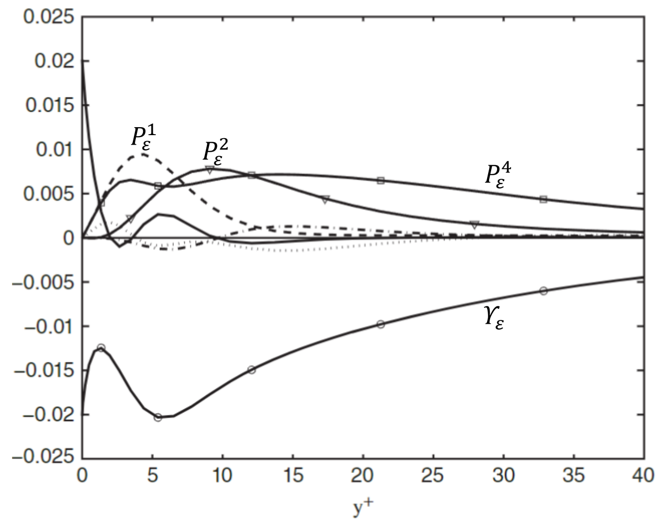


Figure 7.10 ϵ equation budget in channel flow at $R_\tau = 590$ [20] scaled with ν and u_τ. −−, P_ϵ^1; ∇, P_ϵ^2; − · −, P_ϵ^3; □, P_ϵ^4; ○, $-\Upsilon_\epsilon$; —, D_ϵ; · · ·, $\Pi_\epsilon + T_\epsilon$.

For $y^+ \leq 25$, P_ε^1 and P_ε^2 larger compared to P_ε^3 and $\Pi_\varepsilon + T_\varepsilon$.

Near wall $-\Upsilon_\varepsilon$ has minimum and at the wall $-\Upsilon_\varepsilon = D_\varepsilon$.

Reynolds Stress Budget

$$\overline{u^2}:\ 0 = -2\overline{uv}\frac{d\overline{U}}{dy} - \varepsilon_{11} - \frac{d\overline{u^2 v}}{dy} + \Pi_{11} + \nu\frac{d^2\overline{u^2}}{dy^2}$$

$$\overline{v^2}:\ 0 = -\varepsilon_{22} - \frac{d\overline{v^3}}{dy} + \Pi_{22} + \nu\frac{d^2\overline{v^2}}{dy^2} - \frac{2}{\rho}\frac{d\overline{pv}}{dy}$$

$$\overline{w^2}:\ 0 = -\varepsilon_{33} - \frac{d\overline{w^2 v}}{dy} + \Pi_{33} + \nu\frac{d^2\overline{w^2}}{dy^2}$$

Where:

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \quad \Pi_{ij} = \frac{1}{\rho}\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} = \frac{1}{\rho}\overline{p2\Sigma_{ij}}$$

$\Sigma_{ij} =$ turbulent rate of strain = $\frac{1}{2}(u_{i,j} + u_{j,i})$

Π_{ij}= pressure strain correlation

$\overline{u^2}$ connected mean flow via production term.

$\overline{v^2}$ and $\overline{w^2}$ production via Π_{22} and Π_{33}.

$\sum \Pi_{ii} = 0$, thus Π_{11} mostly < 0.

Consequently, P in $\overline{u^2}$ is transferred to $\overline{v^2}$ and $\overline{w^2}$ via Π_{ii}.

Near wall $\Pi_{33} \sim \Pi_{11} + \Pi_{22}$

Near channel center:

$\overline{u^2}$: $P \sim \varepsilon_{11} + \Pi_{11}$

$\overline{v^2}$ and $\overline{w^2}$: $\varepsilon \sim \Pi_{22}$ and Π_{22}, respectively

Complex physics $y^+ < 40$

$\overline{u^2}$: $\varepsilon \sim \nu \overline{u^2}_{yy}$

$\overline{w^2}$: $\varepsilon \sim \nu \overline{w^2}_{yy}$

$\overline{v^2}$: $\overline{pv}_y \sim \Pi_{22} \rightarrow \Pi_{33}$

$\overline{v^2}$ and $\overline{w^2}$: $\overline{pv}_y|_{\overline{v^2}\text{ near wall}} \rightarrow \Pi_{33}$

However, net effect $\overline{pv}_y$ and Π_{22} small and mostly cancel.

$\overline{w^2}$: losses due to ε near wall balanced by $\nu \overline{w^2}_{yy}$

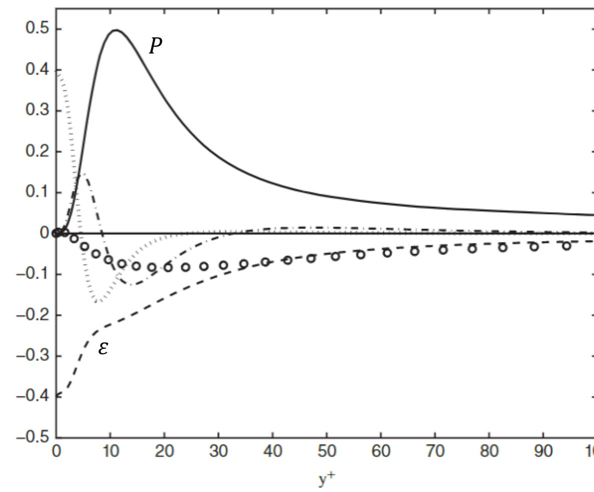


Figure 7.11 $\overline{u^2}$ budget in channel flow for $R_\tau = 5186$ [10]. —, production; – –, dissipation; ○, pressure strain; · · ·, viscous diffusion; – · –, turbulent transport.

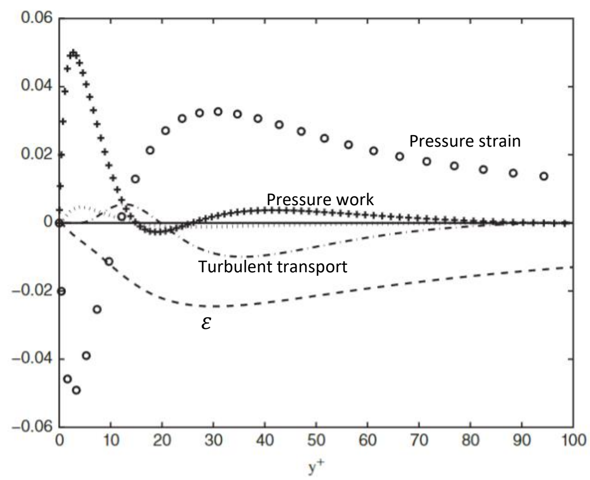


Figure 7.12 $\overline{v^2}$ budget in channel flow for $R_\tau = 5186$ [10]. – –, dissipation; ○, pressure strain; +, pressure work; · · ·, viscous diffusion; – · –, turbulent transport.

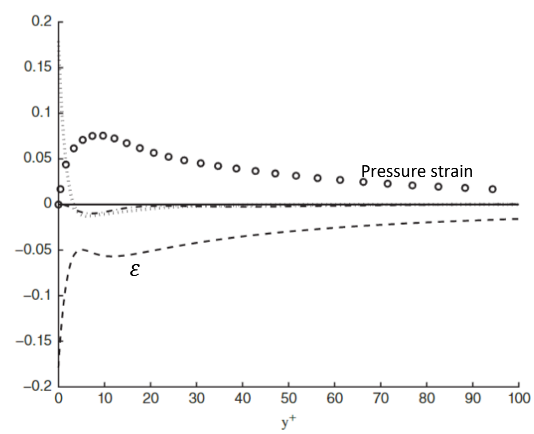


Figure 7.13 $\overline{w^2}$ budget in channel flow for $R_\tau = 5186$ [10]. – –, dissipation; ○, pressure strain; · · ·, viscous diffusion; – · –, turbulent transport.

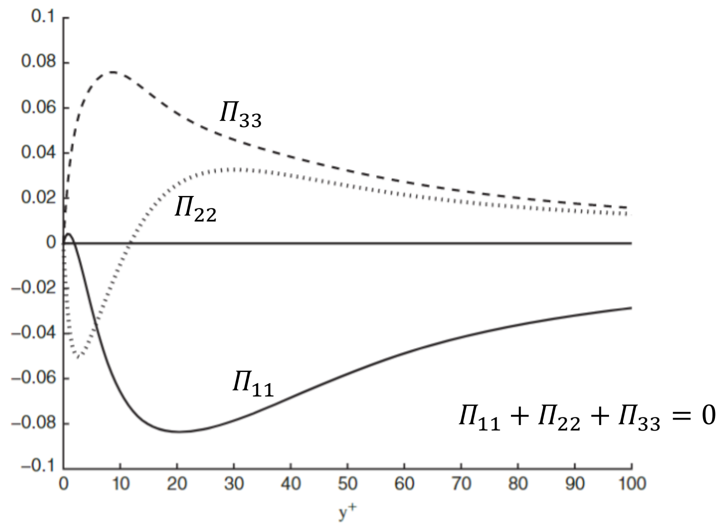


Figure 7.14 Pressure-strain term in normal Reynolds stress equations at $R_\tau = 5186$ [10]: —, Π_{11}; · · ·, Π_{22}; – –, Π_{33}.

Viscous and turbulent transport result in spatial redistribution.

Consider now Reynolds $\overline{uv}$ balance:

$$0 = \underbrace{-\overline{v^2}\frac{d\overline{U}}{dy}}_{\boxed{1}} - \underbrace{\varepsilon_{12}}_{\boxed{2}} + \underbrace{\Pi_{12}}_{\boxed{3}} - \underbrace{\frac{d\overline{uv^2}}{dy}}_{\boxed{4}} - \underbrace{\frac{1}{\rho}\frac{d\overline{pu}}{dy}}_{\boxed{5}} + \underbrace{\nu\frac{d^2\overline{uv}}{dy^2}}_{\boxed{6}}$$

1) Production
2) Dissipation: small
3) Pressure strain
4) Turbulent transport
5) Pressure work
6) Viscous diffusion: small

4) and 5) cancel.

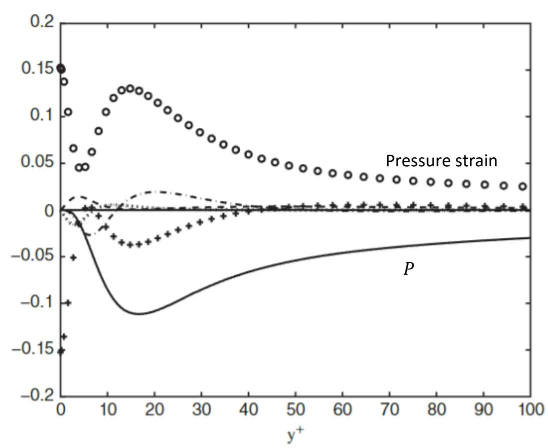


Figure 7.15 $\overline{uv}$ budget in channel flow for $R_\tau = 5186$ [10]. —, production; – –, dissipation; ○, pressure strain; +, pressure work; · · ·, viscous diffusion; – · –, turbulent transport.

$\overline{uv} < 0$ lower channel produced by 1)

Mostly balance of P and Π_{12} → important to model it correctly.

2) and 6) small → $\overline{uv} \neq f(\nu)$

3) and 5) nearly cancel near wall → can be combined for modeling.

Enstrophy budget

$$\zeta = \overline{\omega_1^2} + \overline{\omega_2^2} + \overline{\omega_3^2}$$

$\overline{\omega_1^2} \sim \overline{\omega_2^2} \sim \overline{\omega_3^2}$ away from wall.

$\overline{\omega_1^2}(0) = \overline{w_y^2}(0)$

$\overline{\omega_2^2}(0) = 0$

$\overline{\omega_3^2}(0) = \overline{u_y^2}(0) \gg \overline{w_y^2}(0)$

Also associated $\overline{\Omega_3} = -\overline{U_y}$ near wall Fig. 7.3.

Peak in $\overline{\omega_1^2}(0)$ due to spanwise motions near wall.

Anisotropy $10 \le y^+ \le 30$: complex physics buffer layer

$$\frac{\varepsilon}{\nu} = \zeta + \frac{\mathrm{d}^2\overline{v^2}}{dy^2}$$

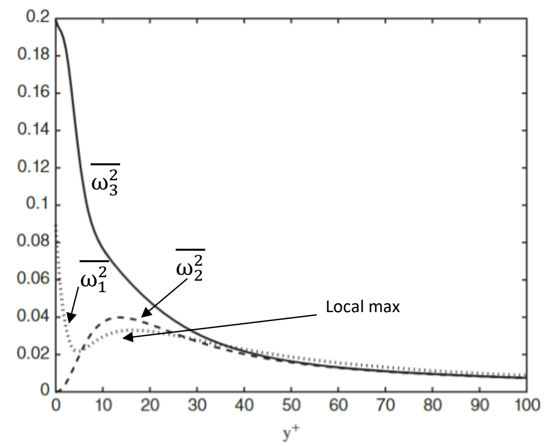


Figure 7.16 Comparison of the enstrophy components in channel flow at $R_\tau = 5186$ [10]. $\cdots$, $\overline{\omega_1^2}^+$; $--$, $\overline{\omega_2^2}^+$; —, $\overline{\omega_3^2}^+$.

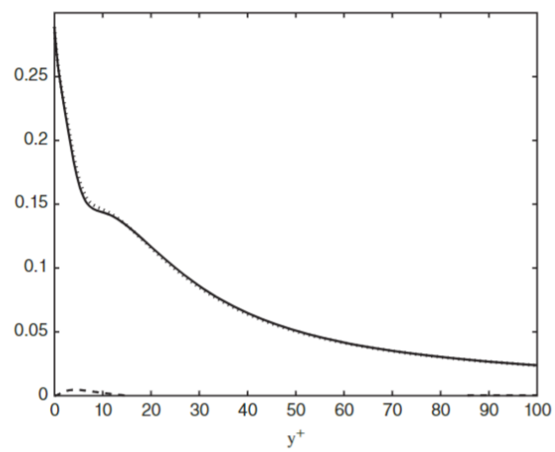


Figure 7.17 Evaluation of the terms in Eq. (7.54) in channel flow with $R_\tau = 5186$ [10]: $\cdots$, $-\epsilon^+$; —, ζ^+; $--$, $d^2\overline{v^2}^+/dy^{+2}$.

Appendix

$\underline{U}(0) = 0$ determines how $\langle u_i u_j \rangle$ depart from zero for small y. For fixed x, z, and t and small y TS for $\underline{u}$:

$$u = a_1 + b_1 y + c_1 y^2 + \cdots$$
$$v = a_2 + b_2 y + c_2 y^2 + \cdots$$
$$w = a_3 + b_3 y + c_3 y^2 + \cdots$$

coefficients are zero mean random variables and for fully developed channel flow statistically independent x, z, and t

no slip: $a_1 = a_2 = a_3 = 0$

$u_x = w_z = 0 \Rightarrow v_y = 0$

ie $b_2 = 0$

$b_2 = 0 \Rightarrow$ 2 component flow close to surface

$\langle u_i u_j \rangle$ obtained by taking means of products of TS:

$$\langle u^2 \rangle = \langle b_1^2 \rangle y^2 + \cdots$$
$$\langle v^2 \rangle = \langle c_2^2 \rangle y^4 + \cdots$$
$$\langle w^2 \rangle = \langle b_3^2 \rangle y^2 + \cdots$$
$$\langle uv \rangle = \langle c_1 c_2 \rangle y^3 + \cdots$$

$\langle u^2 \rangle$, $\langle w^2 \rangle$, [illegible] increase as y^2, whereas $-\langle uv \rangle$ and $\langle v^2 \rangle$ increase more slowly ie y^3 and y^4, respectively.

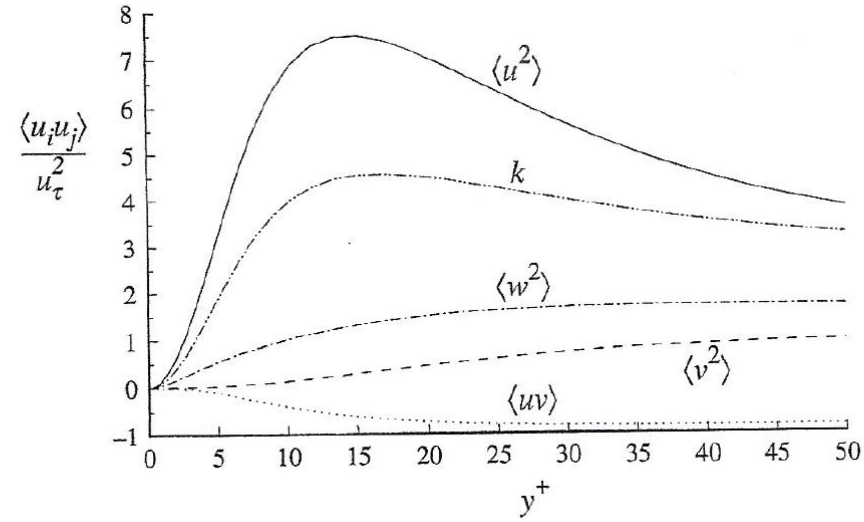


Fig. 7.17. Profiles of Reynolds stresses and kinetic energy normalized by the friction velocity in the viscous wall region of turbulent channel flow: DNS data of Kim *et al.* (1987). Re = 13,750.

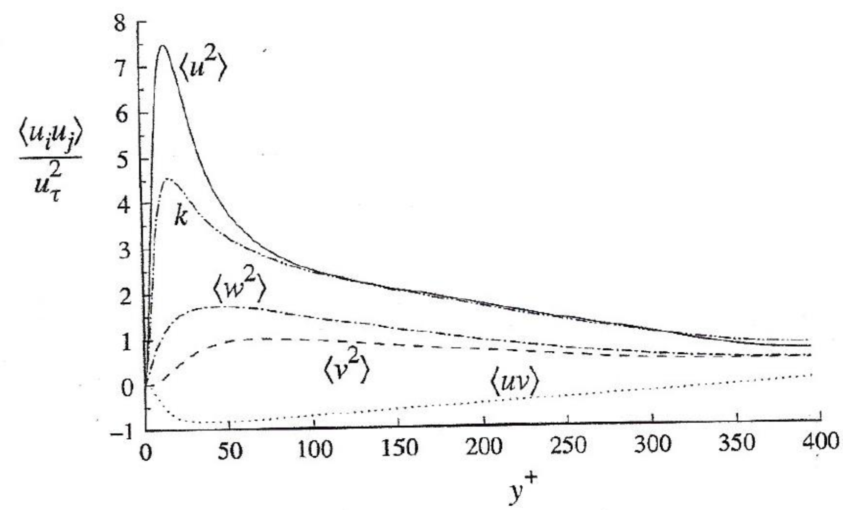


Fig. 7.14. Reynolds stresses and kinetic energy normalized by the friction velocity against y^+ from DNS of channel flow at Re = 13,750 (Kim *et al.* 1987).