Chapter 7: Properties of Turbulent Free Shear Flow (Chap. 11 Bernard)

Part 2: Turbulent Wake: Circular Cylinder

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Large-eddy simulation of the flow past a circular cylinder at sub- to super-critical Reynolds numbers

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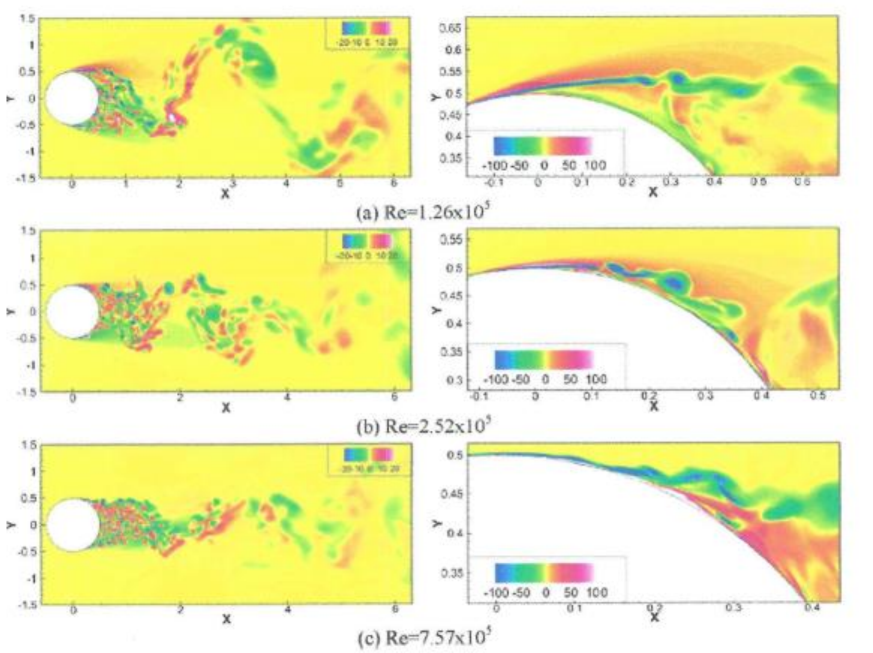
ABSTRACT

Large-eddy simulation of turbulent flow past a circular cylinder at sub- to super-critical Reynolds numbers is performed using a high-fidelity orthogonal curvilinear grid solver. Verification studies investigate the effects of grid resolution, aspect ratio and convection scheme. Monotonic convergence is achieved in grid convergence studies. Validation studies use all available experimental benchmark data. Although the grids are relatively large and fine enough for sufficiently resolved turbulence near the cylinder, the grid uncertainties are large indicating the need for even finer grids. Large aspect ratio is required for sub-critical Reynolds number cases, whereas small aspect ratio is sufficient for critical and super-critical Reynolds number cases. All the experimental trends were predicted with reasonable accuracy, in consideration the large facility bias, age of most of the data, and differences between experimental and computational setup in particular free stream turbulence and roughness. The largest errors were for under prediction of turbulence separation.



(a) $Re=1.26x10^5$

(b) $Re=2.52x10^5$

(c) $Re=7.57x10^5$

Fig. 7. Instantaneous spanwise vorticity contours, right side shows the close-up views.

Self-Preserving Far Wake

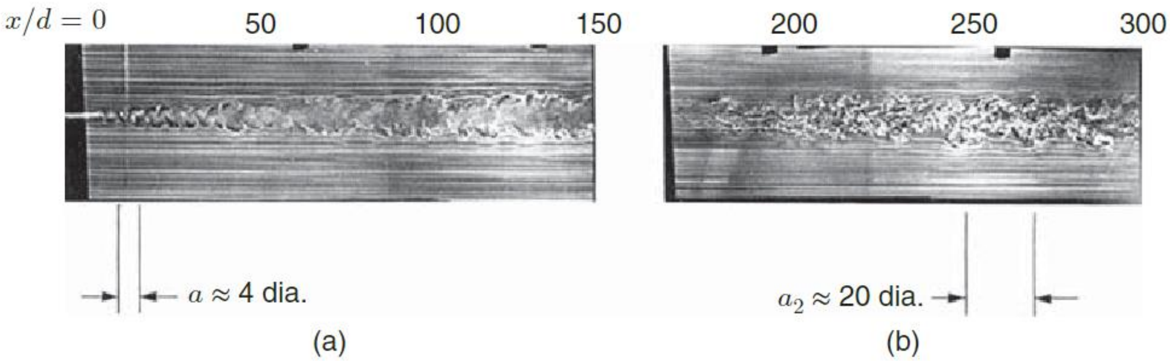


Figure 11.2 Circular cylinder wake at $R_e = 2200$; smoke wire at (a) $x/d = 1$ and (b) $x/d = 160$, [3]. Reprinted with permission of Cambridge University Press.

Near wake: organized Karman vortices.

Far wake: disorganized large-scale vortices/broad-band and no dominant frequencies.

For far downstream wake flow, conditions for similarity solution of mean velocity achieved → diffrerences between mean velocity profiles at different x locations attributable to change in scale, not in functional form.

$$\overline{U} = U_e - \underbrace{\Delta U f(\eta)}_{\boxed{\text{velocity defect}}} \qquad (1)$$

$$-\overline{uv} = (\Delta U)^2 g(\eta) \qquad (2)$$

Where:

$$\eta = \frac{y}{l(x)}$$

is a similarity variable and $\Delta U(x) = U_e - U_{min}(x)$. Velocity defect obeys similarity law, which by the definition of ΔU, $f(0) = 1$, while symmetry implies that $f'(0) = 0$. Anti-symmetry in RS implies that $g(0) = 0$.

BC1: $\overline{U}(x,0) = U_{min}(x) = \overline{U_{min}}$

$$\overline{U}(x,0) = U_e - \Delta U f(0)$$

$$= U_e - \left(U_e - U_{min}(x)\right) f(0)$$

$$f(0) = 1$$

BC2: $d\overline{U}(x,0)/dy = 0$

$$\frac{d\cancel{U_e}}{\cancel{dy}} - \frac{d\cancel{\Delta U}}{\cancel{dy}} f(0) - \Delta U \underbrace{\frac{df(0)}{d\eta}}_{\boxed{f'(0)}} \frac{d\eta}{dy} = 0$$

$$-\Delta U f'(0) \frac{d\left(\frac{y}{l}\right)}{dy} = -\frac{\Delta U}{l} f'(0) = 0$$

$$f'(0) = 0$$

Idea is to use momentum equation:

$$\frac{\partial}{\partial x}\left[\overline{U}\left(\overline{U}-U_e\right)\right]+\frac{\partial}{\partial y}\left[\overline{V}\left(\overline{U}-U_e\right)\right]+\frac{\partial}{\partial y}\overline{uv}=0 \quad (3)$$

to explore nature of similarity solutions.

Integrating mean continuity equation ($\partial\overline{U}/\partial x\ +\partial\overline{V}/\partial y=0$) :

$$\overline{V}=-\int_0^y \frac{\partial\overline{U}}{\partial x}dy \quad (4)$$

Since $\overline{V}=0$ at centerline $y=0$.

Differentiating Eq. (1) with respect to x gives

$$\frac{\partial\overline{U}}{\partial x}=-f\frac{d\Delta U}{dx}-\Delta U\underbrace{\frac{df}{d\eta}}_{\boxed{f'}}\underbrace{\frac{d\eta}{dl}}_{\boxed{-\frac{\eta}{l}}}\frac{dl}{dx}$$

$$\frac{\partial\overline{U}}{\partial x}=-f\frac{d\Delta U}{dx}+\Delta U f'\frac{\eta}{l}\frac{dl}{dx} \quad (5)$$

Differentiating Eq. (1) with respect to y yields:

$$\frac{\partial\overline{U}}{\partial y}=\cancel{\frac{\partial U_e}{\partial y}}-\Delta U\frac{\partial f(\eta)}{\partial y}=-\Delta U\underbrace{\frac{df}{d\eta}}_{\boxed{f'}}\underbrace{\frac{d\eta}{dy}}_{\boxed{\frac{1}{l}}}$$

$$\frac{\partial\overline{U}}{\partial y}=-\frac{\Delta U}{l}f' \quad (6)$$

Differentiating Eq. (2) with respect to y gives:

$$\frac{\partial \overline{uv}}{\partial y} = -(\Delta U)^2 \frac{\partial g(\eta)}{\partial y} = -(\Delta U)^2 \underbrace{\frac{dg}{d\eta}}_{\boxed{g'}} \underbrace{\frac{d\eta}{dy}}_{\boxed{\frac{1}{l}}}$$

$$\frac{\partial \overline{uv}}{\partial y} = -\frac{\Delta U^2}{l} g' \quad (7)$$

Substituting Eq. (5) into (4) and converting the y integration into η integration gives

$$\overline{V} = -\int_0^\eta \left[-f \frac{d\Delta U}{dx} + \Delta U f' \frac{\eta}{l} \frac{dl}{dx}\right] l d\eta \qquad \boxed{dy = l d\eta}$$

$$= l \frac{d\Delta U}{dx} \underbrace{\int_0^\eta f d\eta}_{\boxed{G(\eta)}} - \Delta U \frac{dl}{dx} \underbrace{\int_0^\eta f' \eta d\eta}_{\boxed{H(\eta)}}$$

$$\overline{V} = l \frac{d\Delta U}{dx} G(\eta) - \Delta U \frac{dl}{dx} H(\eta) \quad (8)$$

Using Eqs. (5), (6), (7) and (8) and dividing by $\Delta U^2/l$, Eq. (3) becomes:

$$-\alpha^* f + \beta^* \eta f' + \alpha^* \frac{\Delta U}{U_e}[-f'G + f^2] - \beta^* \frac{\Delta U}{U_e}[-f'H + \eta f f'] = g' \quad (9)$$

Appendix A.1

Where:

$$\alpha^* = \frac{U_e l}{(\Delta U)^2} \frac{d\Delta U}{dx} \quad (10)$$

And

$$\beta^* = \frac{U_e}{\Delta U} \frac{dl}{dx} \quad (11)$$

Represent dimensionless parameters.

Eq. (9) shows that sufficient condition for a similarity solution to exist is that α^* and $\beta^* \neq f(x)$.

In far wake

$$\frac{\Delta U}{U_e} \rightarrow 0 \;\; \text{as} \;\; x \rightarrow \infty$$

Such that Eq. (15) simplifies to:

$$-\alpha^* f + \beta^* \eta f' = g'$$

Taking the ratio of Eqs. (10) and (11) gives:

$$\frac{\alpha^*}{\beta^*} = \frac{\frac{\Delta U_x}{\Delta U}}{\frac{l_x}{l}} \equiv n = \text{constant}$$

Assuming the rate of growth of $l(x)$ and the rate of decay of $\Delta U(x)$ are equal, then their ratio is a constant. Consequently,

$$\frac{\Delta U_x}{\Delta U} = \frac{l_x}{l} n$$

$$\ln \Delta U = n \ln l + C = \ln l^n + C$$

$$\Delta U = C l^n \quad (12)$$

Substituting Eq. (12) into (11), gives

$$\beta^* = \frac{U_e}{\Delta U} \frac{dl}{dx} = \frac{U_e}{C l^n} \frac{dl}{dx}$$

$$\frac{\beta^* C}{U_e} dx = \frac{dl}{l^n} \quad (13)$$

Integrating Eq. (13) yields:

$$\frac{\beta^* C}{U_e}(x - x_0) = \frac{l^{1-n}}{(1-n)}$$

$$\underbrace{(1-n)\frac{\beta^* C}{U_e}}_{\boxed{\alpha}}(x - x_0) = l^{1-n} \qquad \boxed{\alpha = (1-n)\beta^* C/U_e}$$

$$l(x) = \alpha^m (x - x_0)^m \quad (14)$$

Where x_0 = virtual origin and

$$m = \frac{1}{1-n}$$

Substituting Eq. (14) into (12) gives

$$\Delta U(x) = C\alpha^{m-1}(x - x_0)^{m-1} \quad (15)$$

Recall definition of total mean flux of momentum per unit length in spanwise direction:

$$M = \rho \int_{-\infty}^{\infty} \overline{U}(\overline{U} - U_e)dy = \text{constant} \neq f(x)$$

$$M = -\rho U_e^2 \theta = -\text{body drag, which induces wake}$$

Where:

$$\theta = \int_{-\infty}^{\infty} \frac{\overline{U}}{U_e}\left(1 - \frac{\overline{U}}{U_e}\right)dy \quad (16)$$

Represents the momentum thickness, in analogy to boundary layer theory, and it is constant in wake flow.

Substituting $\overline{U} = U_e - \Delta U f(\eta)$ into Eq. (16) gives:

$$\theta = \int_{-\infty}^{\infty} \frac{U_e - \Delta U f(\eta)}{U_e}\left(1 - \frac{U_e - \Delta U f(\eta)}{U_e}\right) dy$$

$$\theta = \int_{-\infty}^{\infty} \left(1 - \frac{\Delta U f(\eta)}{U_e}\right)\left(\cancel{1} - \cancel{1} + \frac{\Delta U f(\eta)}{U_e}\right) \underbrace{dy}_{\boxed{l d\eta}}$$

$$\theta = \frac{\Delta U}{U_e}\int_{-\infty}^{\infty} \left(1 - \frac{\Delta U f(\eta)}{U_e}\right) f(\eta) l \, d\eta$$

Dividing by l:

$$\frac{\theta}{l} = \frac{\Delta U}{U_e}\left[\int_{-\infty}^{\infty} f(\eta)\, d\eta - \underbrace{\cancel{\frac{\Delta U}{U_e}\int_{-\infty}^{\infty} f^2(\eta)\, d\eta}}_{\frac{\Delta U}{U_e} \to 0 \text{ far wake}}\right]$$

Therefore,

$$l\Delta U = \frac{U_e \theta}{\int_{-\infty}^{\infty} f(\eta)\, d\eta} = \text{constant} \neq f(x)$$

$\therefore$ $l\Delta U \neq f(x)$ and equal to a constant in the far wake, as previously assumed, i.e., assumption n = constant.

Substituting Eqs. (14) and (15) for $l\Delta U$ gives:

$$l(x)\Delta U(x) = \alpha^m (x - x_0)^m \, C\alpha^{m-1}(x - x_0)^{m-1} \neq f(x) \quad (17)$$

i.e., $m + m - 1 = 0 \rightarrow m = 1/2$, such that:

$$l(x) = \alpha^{1/2}(x - x_0)^{1/2} \quad (18)$$

$$\Delta U(x) = C\alpha^{-1/2}(x - x_0)^{-1/2} \quad (19)$$

Circular cylinder reaches self-similarity about 80-90 diameters downstream for mean variables and larger distance for turbulence variables.

Using control volume analysis, a relationship between θ and drag (D) can be established (Betz Method):

$$D = \rho U_e^2 \int_{-\infty}^{\infty} \frac{\overline{U}}{U_e}\left(1 - \frac{\overline{U}}{U_e}\right) dy = \rho U_e^2 \theta$$

Now, that relations for $l(x)$ and $\Delta U(x)$ are established, it is possible to find the mean velocity field $\overline{U}$, by determining $f(\eta)$ via

$$-\alpha^* f + \beta^* \eta f' = g' \quad (20)$$

once a model for $g(\eta) = -\overline{uv}/(\Delta U)^2$ is proposed.

Traditional approach → eddy viscosity model with ν_t = constant.

$$\overline{uv} = -\nu_t \frac{\partial \overline{U}}{\partial y} = -(\Delta U)^2\, g(\eta) \quad (21)$$

Recall

$$\frac{\partial \overline{U}}{\partial y} = -\frac{\Delta U}{l} f'$$

And substituting into Eq. (21) gives

$$\nu_t \frac{\Delta U}{l} f' = -(\Delta U)^2\, g(\eta)$$

$$g(\eta) = -\nu_t \frac{f'}{l \Delta U} = -\frac{f'}{R_t} \quad (22)$$

Where:

$$R_t = \frac{l\Delta U}{\nu_t}$$

Is a constant Reynolds number; since $l\Delta U =$ constant.

Substituting Eq. (22) into (20) gives

$$-\alpha^* f + \beta^* \eta f' = -\frac{f''}{R_t} \quad (23)$$

Moreover, recall

$$m = \frac{1}{1-n} = \frac{1}{2} \rightarrow n = -1$$

And

$$n = \frac{\alpha^*}{\beta^*} \rightarrow \beta^* = -\alpha^*$$

Consequently, Eq. (23) can be rewritten as

$$-\alpha^*(f + \eta f') = -\frac{f''}{R_t}$$

$$f'' - R_t \alpha^*(f + \eta f') = 0 \quad (24)$$

Rewriting Eq. (24) as

$$f'' - R_t \alpha^* \frac{d}{d\eta}(\eta f) = 0$$

$$\boxed{f + \eta f' = \frac{d}{d\eta}(\eta f)}$$

And integrating with respect to η gives:

$$\frac{df}{d\eta} - R_t \alpha^*(\eta f) = C$$

Where $C = 0$, due to BC $f'(0) = 0$

$$\frac{df}{f} = R_t \alpha^* \eta d\eta$$

Integrating again with respect to η

$$\ln f = R_t \alpha^* \frac{\eta^2}{2} + C$$

$$f(\eta) = Ce^{\alpha^* \frac{\eta^2}{2} R_t}$$

Where $C = 1$, due to BC $f(0) = 1$.

Substituting the definitions of $\alpha^* = \frac{U_e l}{(\Delta U)^2} \frac{d\Delta U}{dx}$ and $R_t = \frac{l\Delta U}{\nu_t}$ combined with Eqs. (18) and (19) gives:

$$f(\eta) = \exp\left(\frac{U_e l^2}{\Delta U \nu_t} \frac{d\Delta U}{dx} \frac{\eta^2}{2}\right)$$

$$= \exp\left[-\frac{\cancel{C}}{2} \frac{U_e \cancel{\alpha} \cancel{(x - x_0)}}{\cancel{C} \cancel{\alpha^{-1/2}} \cancel{(x - x_0)^{-1/2}} \nu_t} \cancel{\alpha^{-1/2}} \cancel{(x - x_0)^{-3/2}} \frac{\eta^2}{2}\right]$$

$$f(\eta) = e^{-\frac{U_e \alpha}{4\nu_t} \eta^2}$$

α only effects scaling of distances → can be chosen arbitrarily → $\alpha = d$, such that:

$$f(\eta) = e^{-\frac{R_d \eta^2}{4}}$$

Is a Gaussian function where:

$$R_d = \frac{U_e d}{\nu_t}, \text{ i.e., } \nu_t = \frac{U_e d}{R_d}$$

And

$$\eta = \frac{y}{\sqrt{d(x - x_0)}}$$

Experimental measurements of

$$\frac{U_e - \overline{U}}{\Delta U} = f(\eta)$$

In the far wake of a circular cylinder at several cross sections are shown in Fig. 11.3.

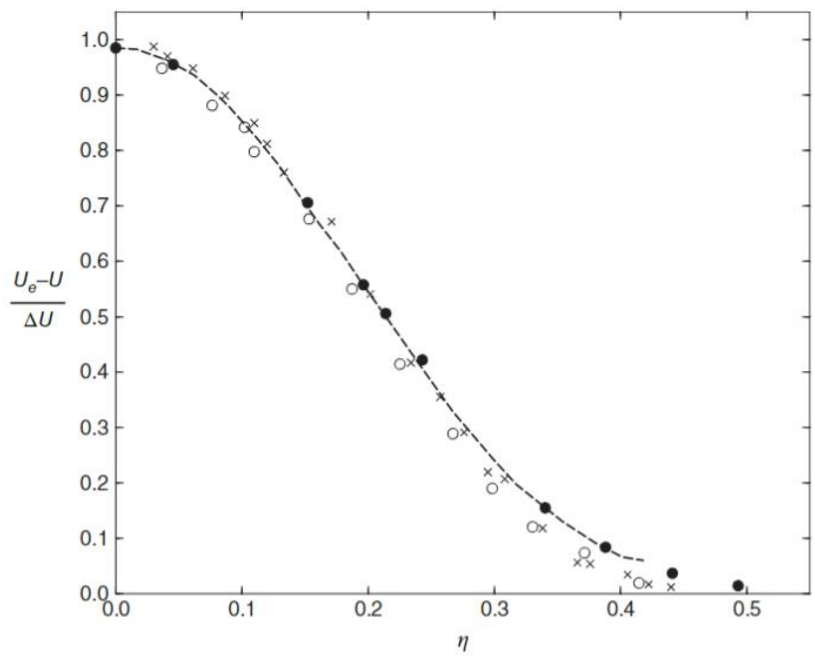


Figure 11.3 Comparison of the self-similar turbulent wake velocity profile of a cylinder with physical experiments at $R_d = 1360$: •, $x/d = 500$; +, $x/d = 650$; ○, $x/d = 800$; ×, $x/d = 950$; —, Eq. (11.44). Data from [9]. Reproduced from the *Australian Journal of Scientific Research* (Vol. A2, 1949), with permission of CSIRO Publishing.

$\eta < 0.3$ good fit using $R_d = 61.04$.

Outer part discrepancies due to using $\nu_t =$ constant and intermittency. Including an intermittency factor $\gamma(\eta)$ shows better agreement.

Integrating

$$l\Delta U = \frac{U_e\theta}{\int_{-\infty}^{\infty} f(\eta)\, d\eta} = C$$

Using

$$f(\eta) = e^{-\frac{R_d\eta^2}{4}}$$

Gives

$$C = \sqrt{\frac{R_d}{\pi}}\frac{U_e\theta}{2} = 2.204U_e\theta$$

Such that

$$\frac{\Delta U(x)}{U_e} = 2.204\frac{\theta}{d}\sqrt{\frac{d}{x-x_0}}$$

Introducing the drag coefficient

$$c_D \equiv \frac{D}{\frac{1}{2}\rho d U_e^2}$$

And recalling that

$$D = \rho U_e^2\theta$$

It is found that:

$$\frac{\theta}{d} = \frac{1}{2}c_D$$

Such that:

$$\frac{\Delta U(x)}{U_e} = \frac{2.204c_D}{2}\sqrt{\frac{d}{x-x_0}}$$

Appendix A

A.1

$$\frac{\partial}{\partial x}\left[\overline{U}\left(\overline{U}-U_e\right)\right]+\frac{\partial}{\partial y}\left[\overline{V}\left(\overline{U}-U_e\right)\right]+\frac{\partial}{\partial y}\overline{uv}=0 \quad (1A)$$

Recall

$$\overline{U}=U_e-\underbrace{\Delta U f(\eta)}_{\boxed{\text{velocity defect}}} \quad (2A)$$

$$\overline{V}=l\frac{d\Delta U}{dx}G(\eta)-\Delta U\frac{dl}{dx}H(\eta) \quad (3A)$$

$$-\overline{uv}=(\Delta U)^2 g(\eta) \quad (4A)$$

$$\frac{\partial\overline{uv}}{\partial y}=-\frac{\Delta U^2}{l}g' \quad (5A)$$

Differentiating Eq. (2A) with respect to x:

$$\frac{\partial}{\partial x}\left(\overline{U}-U_e\right)=-f\frac{d\Delta U}{dx}+\Delta U f'\frac{\eta}{l}\frac{dl}{dx} \quad (6A)$$

Differentiating Eq. (2A) with respect to y:

$$\frac{\partial}{\partial y}\left(\overline{U}-U_e\right)=-\frac{\Delta U}{l}f' \quad (7A)$$

Expanding the derivatives in Eq. (1A) yields:

$$\frac{\partial\overline{U}}{\partial x}\left(\overline{U}-U_e\right)+\overline{U}\frac{\partial}{\partial x}\left(\overline{U}-U_e\right)+\frac{\partial\overline{V}}{\partial y}\left(\overline{U}-U_e\right)+\overline{V}\frac{\partial}{\partial y}\left(\overline{U}-U_e\right)+\frac{\partial}{\partial y}\overline{uv}=0 \quad (8A)$$

Substituting Eqs. (3A), (5A), (6A) and (7A) into (8A) gives:

$$\frac{\partial \overline{U}}{\partial x}\left(\overline{U}-U_e\right)+\overline{U}\left(-f\frac{d\Delta U}{dx}+\Delta U f'\frac{\eta}{l}\frac{dl}{dx}\right)+\frac{\partial \overline{V}}{\partial y}\left(\overline{U}-U_e\right) + \left(l\frac{d\Delta U}{dx}G(\eta)-\Delta U\frac{dl}{dx}H(\eta)\right)\left(-\frac{\Delta U}{l}f'\right)-\frac{\Delta U^2}{l}g' = 0 \quad (9A)$$

Using continuity:

$$\frac{\partial \overline{U}}{\partial x}+\frac{\partial \overline{V}}{\partial y}=0 \rightarrow \frac{\partial \overline{U}}{\partial x}=-\frac{\partial \overline{V}}{\partial y}$$

Such that Eq. (9A) simplifies to:

$$-\overline{U}f\frac{d\Delta U}{dx}+\overline{U}\Delta U f'\frac{\eta}{l}\frac{dl}{dx}-\Delta U f'\frac{d\Delta U}{dx}G(\eta)+\frac{\Delta U^2}{l}f'\frac{dl}{dx}H(\eta)-\frac{\Delta U^2}{l}g' = 0 \quad (10A)$$

Using Eq. (2A), $\overline{U}$ can be substituted by $U_e - \Delta U f(\eta)$ such that Eq. (10A) becomes:

$$(-U_e+\Delta U f)f\frac{d\Delta U}{dx}+(U_e-\Delta U f)\Delta U f'\frac{\eta}{l}\frac{dl}{dx}-\Delta U f'\frac{d\Delta U}{dx}G(\eta) + \frac{\Delta U^2}{l}f'\frac{dl}{dx}H(\eta)-\frac{\Delta U^2}{l}g' = 0 \quad (11A)$$

Dividing Eq. (11A) by $\Delta U^2/l$ gives

$$\frac{-U_e l f}{\Delta U^2}\frac{d\Delta U}{dx}+\frac{f^2 l}{\Delta U}\frac{d\Delta U}{dx}+\frac{U_e\eta f'}{\Delta U}\frac{dl}{dx}-ff'\eta\frac{dl}{dx}-\frac{lf'}{\Delta U}\frac{d\Delta U}{dx}G(\eta)+f'\frac{dl}{dx}H(\eta) = g' \quad (12A)$$

Defining

$$\alpha^* = \frac{U_e l}{(\Delta U)^2}\frac{d\Delta U}{dx} \quad (13A)$$

And

$$\beta^* = \frac{U_e}{\Delta U}\frac{dl}{dx} \quad (14A)$$

Eq. (12A) becomes:

$$-\alpha^* f + \alpha^* \frac{\Delta U}{U_e} f^2 + \beta^* \eta f' - \beta^* \frac{\Delta U}{U_e} f f' \eta - \alpha^* \frac{\Delta U}{U_e} f' G(\eta) + \beta^* \frac{\Delta U}{U_e} f' H(\eta) = g'$$

$$-\alpha^* f + \beta^* \eta f' + \alpha^* \frac{\Delta U}{U_e}\left(f^2 - f' G(\eta)\right) - \beta^* \frac{\Delta U}{U_e}\left(f f' \eta - f' H(\eta)\right) = g'$$