

5 Energy Decay in Isotropic Turbulence

Decay process of TKE by viscous dissipation is ideally studied for homogeneous isotropic turbulence; since, contains all essential physics while yielding equations in their simplest forms. The results are used in many turbulence models, which are applied to general flows.

5.1 Energy Decay

Idealized problem: large region far from boundaries or box with periodic boundary conditions. In either case region/box large enough BC does not influence core flow at $\bar{u}(r)$ and $k(r)$ decay to zero with r well within the domain.

Concept is that a complete statistical description of a homogeneous isotropic turbulence is specified in the region/box of interest; at the GDE we solved for the calculation of the energy decay via the solution of the K and ϵ equations.

For homogeneous isotropic turbulence:-

$$\frac{dK}{dt} = -\epsilon \quad \text{where } \epsilon = \bar{\epsilon} = \overline{v u_{i,j} u_{i,j}} \quad (1)$$

$$\frac{d\epsilon}{dt} = P_\epsilon^4 - Y_\epsilon = -2\nu \overline{u_{i,2} u_{i,j} u_{i,j}} - 2\nu^2 \overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} \right)^2} \quad (2a)$$

Since $\epsilon = \nu S$

$$\frac{d\epsilon}{dt} = \nu P_\epsilon^4 - \nu Y_\epsilon = 2\nu \overline{u_i u_j \frac{\partial u_i}{\partial x_j}} - 2\nu^2 \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \quad (2b)$$

Physics isotropic decay governed by decay K at rate ϵ ; at evolution ϵ due balance RHS (2a) or (2b)

P_ϵ^4 & νP_ϵ^4 : effects vortex stretching > 0
 so represent production of ϵ
 $-Y_\epsilon$ & $-\nu Y_\epsilon$: effects dissipation of dissipation

RHS (2a) can be expressed in terms of R_{ii} & S_{ii} via extension of identity

$$\overline{\frac{\partial u_i}{\partial x_j}(y) u_j(y)} = - \frac{\partial R_{ii}}{\partial x_j} \left(\frac{y}{2} - x \right) \quad (3)$$

$$\overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} \right)^2} = \frac{\partial^4 R_{ii}}{\partial x_j^2 \partial x_j^2} (0)$$

$$\overline{u_{i,2} u_{i,j} u_{i,j}} = \frac{\partial^3 S_{ii}}{\partial x_2 \partial x_j^2} (0)$$

$$\frac{\partial^4 R_{11}}{\partial v_1^2 \partial \xi^2}(\underline{v}) = \bar{u}^2 \left[\frac{24}{v} f'''(v) + 11 f''(v) + v f'(v) \right]$$

$$\text{TS: } f'''(v) = v f''(0) + \frac{v^3}{3!} f^{(4)}(0) + \dots$$

$$\text{Since } f \text{ is even } f(v) \lim_{v \rightarrow 0} \frac{f'''(v)}{v} = f''(0)$$

$$\text{Such that } \frac{\partial^4 R_{11}}{\partial v_1^2 \partial \xi^2}(0) = 35 \bar{u}^2 f''(0)$$

$$\text{Similarly } \frac{\partial^3 S_{11}}{\partial \xi \partial v_1^2}(0) = \frac{35}{2} u_{rms}^3 \xi'''(0)$$

note that $\xi'''(0) < 0$ such that $P_2^4 > 0$
 it represents production ξ via random
 stretching. Thus

$$\frac{dS}{dt} = -35 v u_{rms}^3 \xi'''(0) - 70 v^2 \bar{u}^2 f''(0)$$

ie only depends on two time dependent
 scalars along with K and ξ

Similarly or for term RHS (2a), using (3)

$$\overline{(u_x)^3} = u_{rms}^3 \xi'''(0)$$

$$\overline{(u_{xx})^2} = u_{rms}^2 f''(0)$$

$$S_K = - \frac{\overline{(u_x)^3}}{\overline{(u_x)^2}^{3/2}} = \text{skewness of } u_x$$

$$S_K (\overline{u_x})^2^{3/2} = -\overline{u_x^3} = -\overline{u_{rms}^3} R'''(0)$$

$$\overline{u_x^3} = -\overline{u^2} f''(0) \quad \text{where} \quad f''(0) = \varepsilon / (-K \sqrt{\overline{u^2}})$$

$$= \frac{\varepsilon}{K \sqrt{\overline{u^2}}} = \frac{K \sqrt{\overline{u^2}}}{15 \sqrt{\overline{u^2}}} = \overline{u^2} / \lambda_g^2$$

$$R'''(0) = -S_K / \lambda_g^3 = -S_K \left(\frac{\varepsilon}{15 \overline{u^2} \sqrt{\overline{u^2}}} \right)^{3/2}$$

$$G = \frac{\overline{u^2} (\overline{u_x^2})^2}{(\overline{u_x})^2} = \text{parameter difference of } \overline{u_x}$$

$$S''(0) = G / \lambda_g^4 = G \left(\frac{\varepsilon}{15 \overline{u^2} \sqrt{\overline{u^2}}} \right)^2$$

$$\text{so } \frac{d\varepsilon}{dt} = S_K^* R_T^{1/2} \frac{\varepsilon^2}{K} - G^* \frac{\varepsilon^2}{K} \quad (4)$$

$$S_K^* = \frac{7}{3\sqrt{15}} S_K \quad G^* = \frac{7}{15} G \quad R_T = K^2 / \nu \varepsilon$$

along with $\frac{dK}{dt} = -\varepsilon$ we have two equations in 4 unknowns K, ε, S_K^* & G^* so not closed

Initial state needs to be specified i.e. at $t=0$ $K_0, \varepsilon_0, S_{K0}^*$ & G_0^* or instead S_{K0}^* & G_0^* can specify $f(r)$ & $\Delta(r)$

5.1.1 Turbulent Reynolds Number

$$R_T = \frac{K^2}{\nu \epsilon} = \text{turbulent Re} = \frac{\sqrt{K} K^{3/2} / \epsilon}{\nu}$$

velocity scale $u = \sqrt{K}$ & length scale $l = K^{3/2} / \epsilon$
l related eddy turn over time:

$$T_\epsilon = K / \epsilon \text{ which shows that}$$

$$T_\epsilon^{-1} = \frac{\epsilon}{K} = -\frac{1}{K} \frac{dK}{dt} = \text{fractional rate of energy dissipation}$$

∴ T_ϵ = time scale of TKE dissipation

alternatively,

$$R_T = (K / \epsilon) / (\nu / K) = \text{ratio of turbulent \& viscous time scales} \\ = T_\epsilon / T_\nu$$

$$T_\nu = \nu / K = \text{time scale of viscous dissipation}$$

Large R_T = very energetic turbulence

Small R_T = energy in dissipation range
since rate flow energy drops
= rate at which energy dissipated
= weak turbulence

$R_T \rightarrow 0$ during decay of isotropic turbulence

Since K_T is in sketchy form of $\frac{dL}{dt}$ equation indicates that sketchy important for energetic turbulence or dissipative range

Recall another turbulence $Re = Re_\lambda = \frac{u_{rms} \lambda}{\nu}$
where $\lambda = \lambda_g \propto \lambda_g$

$$\begin{aligned} \text{So } R_T &= \frac{3}{20} R_\lambda^2 \quad \text{using } \lambda = \frac{30 \nu u^2}{\lambda^2} = \frac{15 \nu u^2}{\lambda^2} \\ &= \frac{K^2 \lambda_g^2}{\nu 15 \nu u^2} = \frac{K^2 \lambda_g^2}{\nu^2 15 u^2} = \frac{9 u^2 \lambda_g^2}{4 \nu^2 15 u^2} = \frac{3}{20} R_\lambda^2 \end{aligned}$$

R_T or R_λ can be used to characterize degree of turbulence for homogeneous flow:
 $R_\lambda > 100$ turbulence not weak
 > 1000 Strong turbulence
 < 1 very weak turbulence
 final period decay
 before recombination

Interest is in decay process from initial state $R_T \gg 1$ until $R_T < 1$

$$R_T = K^2 / v \varepsilon$$

$$\frac{dR_T}{dt} = \frac{2K}{v \varepsilon} \frac{dK}{dt} - \frac{K^2}{v \varepsilon^2} \frac{d\varepsilon}{dt}$$

$$\therefore \frac{dR_T}{dt} = -\frac{2K}{v} - S_K^* \sqrt{R_T} \frac{K}{v} + G^* \frac{K}{v}$$

define time scale $Z(t) = \int_0^t \frac{\varepsilon(t)}{K(t)} dt$ $Z(0) = 0$
 note $Z \rightarrow \infty$ as $t \rightarrow \infty$ $K \text{ and } \varepsilon > 0$

$t = t(Z)$ = inverse mapping

$$\text{e.g. } R_T^*(Z) = R_T(t(Z))$$

$$\text{or } R_T^*(t(Z)) = R_T(t)$$

$$\frac{dR_T}{dt} = \frac{dR_T^*}{dZ} \frac{dZ}{dt} = \frac{\varepsilon}{K} \frac{dR_T^*}{dZ}$$

$$\frac{dR_T^*}{dZ} = R_T^* (G^* - Z - S_K^* \sqrt{R_T^*}) \quad (5)$$

(1) & (4) or alternatively (5) can be used to solve the decay problem

G^* & S_K^* are $f(\varepsilon)$ such that in additional assumptions are required

5.2 Modes of Isotropic Decay

Additional assumptions are required to close the equations derived for the decay of isotropic turbulence.

The interest is in the decay at both the high and low Re limits.

Several theories, but all result in a power law in time for the decay process, i.e.

$$K \sim (t - t_0)^{-n} \quad \text{where } t_0 = \text{virtual origin}$$

n and other statistical quantities depend on the different theories.

- (1) Full or partial similarity: for high Re , $K \sim t^{-1}$
- (2) Saffman: for high Re , $K \sim t^{-6/5}$

Validation requires data for $-n$ and other flow statistics; however, many difficulties obtaining archival EFD at DNS for homogeneous isotropic turbulence: finite domain; anisotropy; and initial conditions.

5.3 Self-Similarity

Since isotropic decay assumes homogeneous conditions over all space there is no externally imposed geometric length scale. For self-similarity $L = L(t)$ such that multi-point velocity correlation functions are independent of time.

More the less, small scale "isotropic" turbulence in high wave number ^{dissipation} range is $f(v, t) \sim f(v/L(t))$. Early research used integral length scale L in scaling self-similarity, but later greater success was found using λ - Taylor micro scale.

For similarity:

$$f(v, t) = \tilde{f}(v/L(t))$$

$$h(v, t) = \tilde{h}(v/L(t))$$

Turbulence decay exhibits complete self-similarity if self-preserving if scaling holds all v or partial/incomplete if only holds for limited range of v or for limited range of R_λ .

Under assumption complete similarity
 the decay becomes solvable albeit
 with some unresolved issues especially
 regarding entire decay process

Resell definition $\lambda_f = \sqrt{2} \lambda_g$

$$\lambda_f^2 = -2 / f''(0)$$

$$\lambda_f^2 / L^2 = -2 / \tilde{f}''(0) \quad \text{where } \tilde{f}''(0) \neq f''(0)$$

$$\infty \quad \lambda_f^2 \propto L^2$$

1 can assume $L = \lambda_g$

Similarly

$$h'''(0) = \tilde{h}'''(0) / \lambda_g^3$$

Such that $-S_k = \tilde{h}'''(0)$ and $G = \tilde{f}'''(0)$

i.e. S_k and G are constants during self

Similar isotropic decay and GDE
 are closed system of equations

$$\frac{dk}{dt} = -\Sigma$$

$$S_k^* \text{ and } G^* \text{ are}$$

constants provided

$$\frac{dk}{dt} = S_k^* R_T^{1/2} \frac{\Sigma^2}{K} - G^* \frac{\Sigma^2}{K}$$

by assigned values or
 deduced from $\tilde{f}$ and $\tilde{h}$

Alternately can solve single equation for $R_T^*(z)$

$$\frac{dR_T^*}{dz} = R_T^* (G_0^* - 2 - S_{K_0}^* \sqrt{R_T^*})$$

from $I < R_T^*(0) > I$ to time when $R_T^* \ll I$

5.3.1 Fixed Point Analysis

Consider solutions for $\frac{dR_T^*}{dz} = 0$ i.e.
 $R_T^* = \text{constant}$, which are attracting solutions in dynamical sense as $R_T^*(0)$ converge towards such solutions as $t \rightarrow \infty$.

Dynamical system theory indicates that even for $R_0^* > S_{K_0}^*$, functions of t fixed point solutions are stable under which implies that the solutions of the K & S equations are not sensitive to small changes in R_0^* & $S_{K_0}^*$ during an epidemic.

Fixed points $R_{T\infty}^*$ are solutions of

$$R_{T\infty}^* (G_0^* - 2 - S_{K_0}^* \sqrt{R_{T\infty}^*}) = 0$$

$$R_{T\infty}^* = 0 \quad \text{or} \quad R_{T\infty}^* = \left[\frac{G_0^* - 2}{S_{K_0}^*} \right]^2$$

Solutions beginning with $I < R_T^*(0)$ will move toward one or the other depending on value of G_0^*

For $G_0^* \leq 2$: $R_{T\infty}^* = 0$.

$$G_0^* > 2: R_{T\infty}^* = \left(\frac{G_0^* - 2}{\frac{2}{5} K_0} \right)^2$$

Since for $G_0^* < 2$ RHS of dR_T^*/dt equation negative, R_T^* must decay to zero, whereas for $G_0^* > 2$ the RHS is positive for $R_T^* < R_{T\infty}^*$ & negative for $R_T^* > R_{T\infty}^*$. In either case the RHS will raise or lower R_T^* until it converges to $R_{T\infty}^*$.

Fixed point solution = equilibrium state of isotropic turbulent decay

Note that for $G_0^* > 2$ decay process can not achieve $R_T^* = 0$ so it is implied that self-similarity through the entire decay process is not feasible, which is likely due to using $L=2$ since λ can not change turbulence on larger scales where anisotropy is important & is more relevant

5.3.2 Final Period of Torus Decay

Fixed point (equilibrium solution) $R_T^* = 0$ is the end point of the decay process in which there is no motion. Near this end point $R_T \ll 1$, $\dot{R}_T \neq 0$ and it is of interest to understand the behaviour in this weakened condition.

$$\frac{dR_T}{dt} = \underbrace{S_K^* R_T^{1/2} \frac{z^2}{K}}_{\text{index stretchy}} - G_0^* \frac{z^2}{K}$$

$$= \left[\frac{S_K^* R_T^{1/2}}{G_0^*} - 1 \right] G_0^* \frac{z^2}{K}$$

for $\frac{S_K^* R_T^{1/2}}{G_0^*} \ll 1$ index stretchy is negligible

$$\text{Low } R_T = \frac{K^2}{V^2} = \frac{3}{20} R_T^2 \quad (R_T = \frac{\pi^2 Z_{rms}^2}{V}) : S_K^* \sim .5$$

$$\frac{.3 \times R_T^{1/2}}{1.4} = .21 \sqrt{R_T}$$

$$S_K^* \sim .3$$

$$G_0^* \sim 3$$

$$G_0^* \sim 1.4 < 2$$

$$= .21 \times .32 = .07 \text{ for } R_T = .1$$

$\therefore$ for $R_T < .1$, ie, vicinity $R_T = 0$

$$\frac{dK}{dt} = -Z \quad \Rightarrow \quad \frac{dZ}{dt} = -G_0^* \frac{Z^2}{K}$$

Solution: $k/k_0 = \left(1 + \frac{z}{\alpha T_{\pm 0}}\right)^{-\alpha}$

$z/z_0 = \left(1 + \frac{z}{\alpha T_{\pm 0}}\right)^{-1-\alpha}$

$\alpha = 1/G_0^* - 1$

where $k_0, z_0, \alpha T_{\pm 0} = k_0/z_0$ are the I.C.

After $\pm$ advances several multiple $T_{\pm 0}$ such that $z/(k T_{\pm 0}) \gg 1$ the solutions simplify

$k \sim z^{-\alpha} \quad z \sim z^{-1-\alpha}$

Since for $G_0^* < 2$ so that $\alpha > 1$ at $k=0$ we have the self-similar solutions $k \sim z^{-\alpha}$ and power laws with exponents $\alpha(G_0^*)$

EFD theory shows f is Gaussian in the final period of decay: $f(r,t) = e^{-v^2/2\lambda_f^2}$

$G = \tilde{f}''(0) = 3; G_0^* = 7/5$

$\alpha = 5/2$ i.e. $k \sim z^{-5/2}$

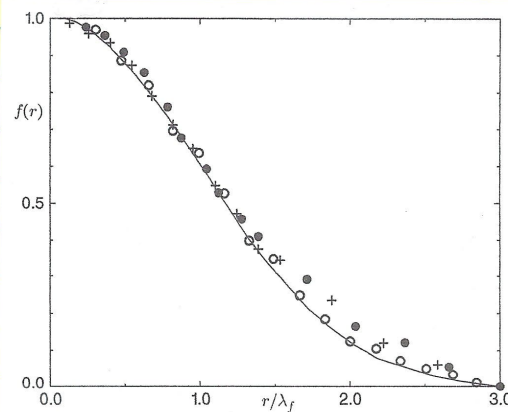


Figure 5.1 Measured and predicted $f(r/\lambda_f)$ in the final period [2]. With permission of the Royal Society.

EFD for grid
turbulence with
mesh spacing M
 $\epsilon = x_1 / T_m$

T_m = mean velocity
 x_1 = distance
along wind tunnel

Linear behaviour $(U_m^2 / \overline{u^2})^{2/5}$
indicates T_m is sufficient
-5/2 decay law is
So small K for isotropic
turbulence

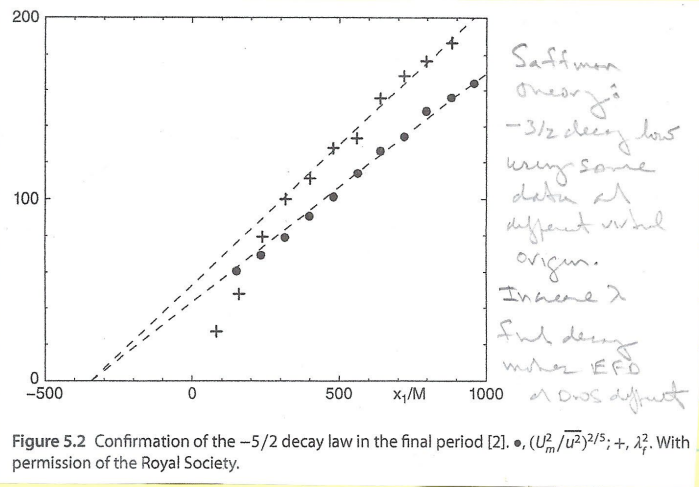


Figure 5.2 Confirmation of the $-5/2$ decay law in the final period [2]. •, $(U_m^2 / \overline{u^2})^{2/5}$; +, λ_f^2 . With permission of the Royal Society.

$$\lambda_f^2 = 10 \nu K / \epsilon \quad \left(\text{from } \epsilon = \frac{15 \nu \overline{u^2}}{\lambda_f^2} \text{ with } K = \frac{3}{2} \overline{u^2} \right)$$

Substituting $K = \epsilon^{-1-\alpha}$ and $\epsilon = \epsilon^{-1-\alpha}$ and $\alpha = 5/2$

$$\lambda_f^2 \sim \epsilon \quad \text{where } \epsilon = \lambda_f \sim \lambda_g$$

$$\alpha = 2 \sim \sqrt{\epsilon} \quad \text{is given by } \sqrt{\epsilon}$$

with explanation that as all scales are
decay faster than large scales which are
causing the scale of the turbulence
turbulence to increase. As the flow
redevelops $f(x) \rightarrow 0$ and $\lambda_f \rightarrow \infty$, as
per Figure where λ_f varies linearly with x_1

Also, since $g = f + \frac{r}{2} f'$

$$g(r, t) = \left(1 - \frac{r^2}{2 \lambda_g^2} \right) e^{-\frac{r^2}{2 \lambda_g^2}} \quad \text{which is } < 0 \text{ for } r > \sqrt{2} \lambda_g$$

For small $R_T \Rightarrow G_0^* < 2$, the $R_{T0}^* = 0$ solution shows reasonable physics; however, this is not the case for large R_T since solutions for $G_0^* < 2$ at R_{T0} large display non-physical results.

$$\frac{dR_T^*}{dt} = R_T^* (G_0^* - 2 - S_{K0}^* \sqrt{R_T^*})$$

$$R_T^* = R_{T0}^* \left[\frac{(G_0^* - 2) \exp((G_0^* - 2)\tau/2)}{G_0^* - 2 - S_{K0}^* \sqrt{R_{T0}^*} (1 - \exp((G_0^* - 2)\tau/2))} \right] \quad G_0^* \neq 2$$

Analytic Solution:

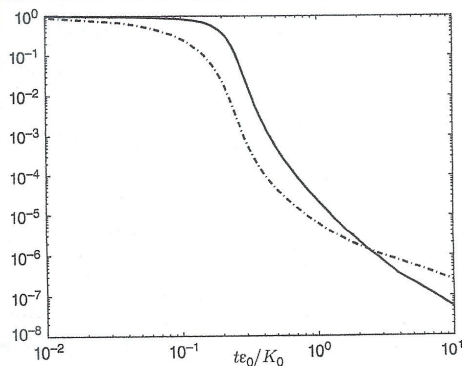
$$= R_{T0}^* \left[\frac{1}{1 + S_{K0}^* \sqrt{R_{T0}^*} \tau/2} \right]^2 \quad G_0^* = 2$$

For $G_0^* < 2$, R_{T0}^* is large and if τ is also large $G_0^* \neq 2$ solution gives

$$R_T^* = e^{(G_0^* - 2)\tau} \text{ i.e. } R_T^* \text{ will quickly decrease}$$

no matter how large R_{T0}^*

$$\text{and since } \tau(t) = \ln(K_0/K(t)) \text{ or } e^\tau = K_0/K(t)$$



unphysical initial drop in K . Time since eddy turnover time $\tau_{E0} = K_0/\varepsilon_0$ such that K drops 5 orders magnitude in one time unit

$$K(t) = K_0/e^\tau \text{ i.e. } K \ll K_0 \text{ almost energy gone}$$

In some time unit R_T falls to order representative final decay so solution unphysical

Figure 5.3 Self-similar decay corresponding to $G_0^* = 7/5$ and $R_{T0}^* = 1000$: —, K/K_0 ; - - -, R_T/R_{T0} .

S.3.3 High Re Equilibrium

For $G_0^* > 2$ R_{T_0} approaches $R_{T_{\infty}}^*$ from above or below depending whether $R_{T_0} <$ or greater $R_{T_{\infty}}^*$

$$\frac{1}{S_k} \frac{d\varepsilon}{dt} = S_k^* R_T^{1/2} \varepsilon^2 / K - G_0^* \varepsilon^2 / K$$

$$\text{Let } S_k^* = G_0^* - 2 / R_T^{1/2}$$

$$\text{i.e. } \frac{d\varepsilon}{dt} = \underbrace{(G_0^* - 2)}_{\text{index stability}} \varepsilon^2 / K - \underbrace{G_0^*}_{\text{dissipation of } \varepsilon} \varepsilon^2 / K$$

$$\frac{d\varepsilon}{dt} = -2 \varepsilon^2 / K \quad \text{from } R_{T_{\infty}}^* = \left(\frac{G_0^* - 2}{S_k^*} \right)^2$$

$R_{T_{\infty}}^*$ large requires large G_0^*
assuming EFO S_k^* values

$R_{T_{\infty}}^* = 10^4$ gives $G_0^* = 25$, which gives near balance vortex stretching & dissipation ε terms $\frac{d\varepsilon}{dt}$ equation, which is an important result. Similar analysis on for $G_0^* < 2$ solution K & ε are asymptotic power laws:

$$K \sim \varepsilon^{-1} \quad \text{and} \quad \varepsilon \sim t^{-2}$$

which is independent ε at different $R_{T_{\infty}}^* = 0$ solution

find solution
 Since λ independent $R_{T_0} \neq R_{T\infty}$, a transient period exists prior to steady $R_{T\infty}$ wherein $SK_0 R_T \rightarrow G_0^* - 2$. The equations show that $R_{T_0} \rightarrow R_{T\infty}$ in very short time i.e. just a few eddy turns over times

Figure shows K, ϵ & R_T
 for $G_0^* = 5$ & $R_{T_0} = 1000$
 with $S_{K_0} = 1.3$. In
 constant low G_0^* stage
 R_T is almost
 $K \rightarrow z^{-1}$ quickly

Long R_T transient period
 K constant & $\epsilon \propto z^{-1}$

$$\lambda_g = [10VK/\epsilon]^{1/2}$$

initially decreases
 then increases due
 insufficient ϵ energy
 in the dissipation
 scales at outset
 simulations

Solutions shows how after initial stage where energy
 from vortex stretching decreases λ_g & ϵ increases, after
 which stretching ~ dissipation & energy decays as per z^{-1}

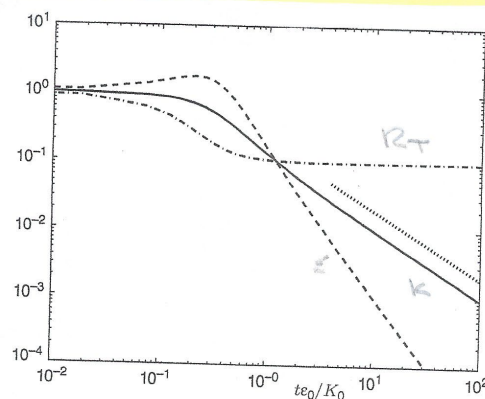
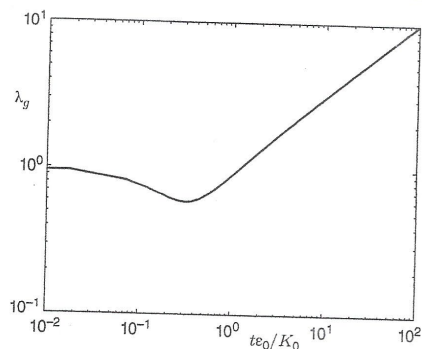


Figure 5.4 Self-similar decay corresponding to $G_0^* = 5$ and $R_{T_0}^* = 1000$: —, K/K_0 ; ---, ϵ/ϵ_0 ; ···, R_T/R_{T_0} ; ···, line of slope -1.

Figure 5.5 λ_g in isotropic decay corresponding to the conditions in Fig. 5.4.



Eventually high Re solution no longer reasonable & self-similarity lost such that G_0^* must decrease to $R_{100}^* = 0$ solution is $G_0^* = 7/5$.

Many EFD wide

range Re

$$1000 \leq Re \leq 5 \times 10^6$$

$$Re = wD/\nu$$

me decay rate

$$-1.18 \pm 0.02$$

Suggests $K = z^{-1.15}$

Subsets suggest other decay rates such as z^{-1} especially for many Re . Many complications determining decay rates from EFD such as z at shifts between decay laws for large & small Re .

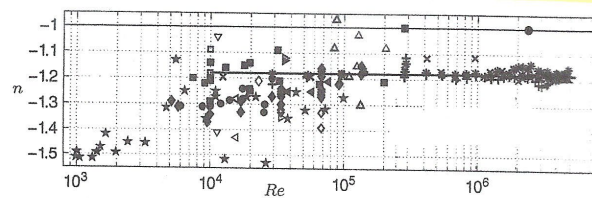


Figure 5.6 Measured power law exponents in decaying homogeneous turbulence from numerous experiments [9]. Filled symbols represent traditional turbulence behind a grid of bars. Open symbols are other turbulence sources. The symbols with $\times$, $*$, and $+$ are from three different probes used in decaying turbulence behind a grid of bars for which the Reynolds number was changed only by altering the viscosity of the working fluid. Used by permission of AIP.

$$\left(\frac{-1.18}{-1.18}\right)^{1/2} = 1.05$$

S.4 Implications for Turbulence Modeling

Isotropic turbulence theory used for modeling ε equation as used for general flows

$$P_{\varepsilon}^4 = S_K^+ R_T^{1/2} \varepsilon^2 / K$$

$$Y_{\varepsilon} = G^* \varepsilon^2 / K$$

As derived for isotropic turbulence

For general application it is assumed that

$$S_K^+ R_T^{1/2} - G^* = -C_{\varepsilon_2} = \text{constant}$$

$$\therefore P_{\varepsilon}^4 - Y_{\varepsilon} = -C_{\varepsilon_2} \varepsilon^2 / K \quad \text{is similar high Re}$$

equilibrium flow

$$\frac{d\varepsilon}{dt} = -2 \varepsilon^2 / K$$

For isotropic turbulence

as per Section 5.1 $Y_{\varepsilon} = \nu Y_{\varepsilon}$

and can be shown using

except constant

$$u - C_{\varepsilon_2}$$

$$\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} \right)^2 = \left(\frac{\partial \omega_i}{\partial x_j} \right)^2$$

$$\text{Next using } \varepsilon = \nu S \Rightarrow G^* \sim \frac{Y_{\varepsilon}/S}{\varepsilon/K} = \frac{\text{fractional rate of change } S}{\text{fractional rate of change } \varepsilon}$$

Since S is function smallest

both are inverse

$$\text{Scales } \varepsilon d = (\nu/\varepsilon)^{1/2} \text{ at small } R_T = \frac{K^2}{\nu \varepsilon}$$

time scales

$$\text{Thus } G^* \sim \sqrt{R_T} \quad \text{for large } R_T$$

Result again for isotropic turbulence

$$\begin{aligned}\frac{d\varepsilon}{dt} &= S_K^+ R_T^{1/2} \frac{\varepsilon^2}{K} - G^+ \frac{\varepsilon^2}{K} \\ &= (S_K^+ R_T^{1/2} - G^+) \frac{\varepsilon^2}{K} \\ &= [S_K^+ R_T^{1/2} - (S_K^+ R_T^{1/2} + C_{\varepsilon_2})] \frac{\varepsilon^2}{K} \\ &= -C_{\varepsilon_2} \frac{\varepsilon^2}{K}\end{aligned}$$

ie coefficient $-C_{\varepsilon_2}$ chosen to cancel vortex stretching term ie the assumption $S_K^+ R_T^{1/2} - G^+ = -C_{\varepsilon_2}$ is equivalent to imposing equilibrium structure on the turbulent decay process, which imposes a decay law

$$K \sim t^{-\frac{1}{C_{\varepsilon_2}-1}} \quad \text{ie } C_{\varepsilon_2} \text{ sets the decay rate}$$

e.g., for $C_{\varepsilon_2} = 11/6$ Saffman $t^{-6/5}$ law recovered. Other values can be achieved via specification C_{ε_2} . In all cases without vortex stretching. If vortex stretching included then eventually t^{-1} decay law will develop.

Another problem is for $v \rightarrow 0$

$$\frac{dz}{dt} = 0 \text{ i.e. } z \text{ is constant}$$

not physical
since z

$$\frac{dz}{dt} = (S_K^* R_T^{1/2} - G^*) z^2 / K \text{ for } v \rightarrow 0$$

should grow
due to vortex

$$\frac{dz}{dt} = S_K^* z^{3/2}$$

$$z = z/v$$

stretching if
 $\frac{dz}{dt} = 0$

$$z(t) = z(0) \left[1 - \frac{S_K^* \sqrt{z(0)} t}{2} \right]^{-2}$$

$$\text{i.e. } z \rightarrow \infty \text{ for } t = 2 / (S_K^* \sqrt{z(0)})$$

This suggests vortex stretching should be
included in $\frac{dz}{dt}$ equation

Grid Turbulence (Pope 5.4.6)

$$\mathbf{U} = U_0 \hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{U} = 0$$

$$P = -\overline{u_i u_j} \frac{\partial \bar{U}_i}{\partial x_j} = 0$$

In reference frame
moving with U_0 the
turbulence is homogeneous
and evolves with $t = x/U_0$

Ideally $\bar{u}_1^2 = \bar{u}_2^2 = \bar{u}_3^2$
 $\bar{u}_i \bar{u}_j = 0 \quad i \neq j$

EFD: $\bar{u}^2 = \bar{v}^2$ and $\bar{u} \bar{v} = 0$
 $\sqrt{\bar{u}^2} \approx 10\% > \sqrt{\bar{v}^2}$

however improvements
possible

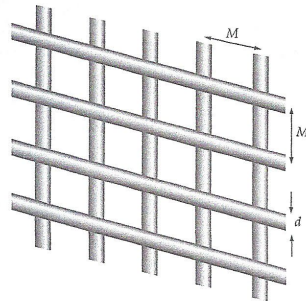


Fig. 5.32. A sketch of a turbulence-generating grid composed of bars of diameter d , with mesh spacing M .

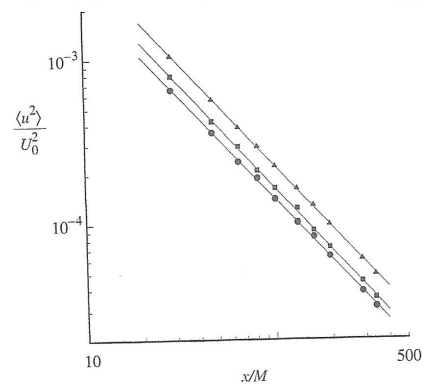


Fig. 5.33. The decay of Reynolds stresses in grid turbulence: squares, $\langle u^2 \rangle/U_0^2$; circles $\langle v^2 \rangle/U_0^2$; triangles k/U_0^2 ; lines, proportional to $(x/M)^{-1.3}$. (From Comte-Bellot and Corrsin (1966).)

$$k/U_0^2 = A \left(\frac{x-x_0}{M} \right)^{-n} \quad x_0 = \text{virtual origin}$$

$$1.15 \leq n \leq 1.45$$

some suggest $n=1.2$ at $x_0=0$

$$A = A(\text{geometry grid}, n, Re)$$

In moving frame

$$k(t) = k_0 \left(t/t_0 \right)^{-n} \quad k_0 = k(0)$$

$$\frac{dk}{dt} = - \left(\frac{nk_0}{t_0} \right) \left(t/t_0 \right)^{-(n+1)}$$

$$\varepsilon(t) = \varepsilon_0 \left(t/t_0 \right)^{-(n+1)} \quad \text{using } \frac{dk}{dt} = -\varepsilon$$

$$\varepsilon_0 = nk_0/t_0$$

Final period of decay:

Re $\downarrow$ such that inertia $\ll$ viscous effects
 Karman-Howarth equation neglects inertia
 term at assuming self similarity
 shows

$$f(v, t) = \exp[-v^2/8\nu t]$$

such that $n = 5/2$.

EXERCISE

5.42 For grid turbulence, given the decay laws for k and ε (Eqs. (5.274) and (5.277), and taking $n = 1.3$), verify the following behaviors:

$$\tau \equiv \frac{k}{\varepsilon} \sim t,$$

$$L \equiv \frac{k^{3/2}}{\varepsilon} \sim \left(\frac{t}{t_0} \right)^{(1-n/2)} = \left(\frac{t}{t_0} \right)^{0.35},$$

$$\frac{k^{1/2}L}{\nu} \sim \left(\frac{t}{t_0} \right)^{(1-n)} = \left(\frac{t}{t_0} \right)^{-0.3}.$$

(Note that the Reynolds number $k^{1/2}L/\nu$ decreases. The increase of L and τ should not be misunderstood. It is not that the turbulent motions become larger and slower. Rather, the smaller, faster motions decay more rapidly, leaving behind the larger, slower motions.)

5.5 Equation for Two-Point Correlations

The analysis of isotropic decay carried out in the previous section concentrates on tracing the history of K and ϵ as they change in time. Only minimal information about

the flow structure was needed, in fact, just the skewness and palenstrophy coefficient that are related to the two-point correlation functions. To proceed to a more extensive analysis of the decay problem that includes analyzing the time dependence of G^* and S_K^* it is necessary to include dynamical information about multi-point correlations. This means introducing an equation for the time history of the two-point velocity correlation tensor $\mathcal{R}(\mathbf{r}, t)$ and then considering its form during isotropic decay. From such an analysis it is also possible to consider the spectral properties of the turbulence during the decay process.

An equation governing $\mathcal{R}_{ij}(\mathbf{x}, \mathbf{y}, t)$ for arbitrary incompressible flow is derived by taking the average of $u_i(\mathbf{x}, t)$ times the j th component of the Navier–Stokes equation in Eq. (2.2) at $\mathbf{y}$ and adding to this the same quantity with i and j and $\mathbf{x}$ and $\mathbf{y}$ reversed. The result is

$$\begin{aligned} & \overline{\rho u_i(\mathbf{x}, t) \frac{\partial U_j}{\partial t}(\mathbf{y}, t) + \rho u_j(\mathbf{y}, t) \frac{\partial U_i}{\partial t}(\mathbf{x}, t)} \\ & + \overline{\rho u_i(\mathbf{x}, t) U_k(\mathbf{y}, t) \frac{\partial U_j}{\partial y_k}(\mathbf{y}, t) + \rho u_j(\mathbf{y}, t) U_k(\mathbf{x}, t) \frac{\partial U_i}{\partial x_k}(\mathbf{x}, t)} = \\ & - \overline{u_i(\mathbf{x}, t) \frac{\partial p}{\partial y_j}(\mathbf{y}, t) - u_j(\mathbf{y}, t) \frac{\partial p}{\partial x_i}(\mathbf{x}, t)} \\ & + \overline{\mu u_i(\mathbf{x}, t) \nabla^2 U_j(\mathbf{y}, t) + \mu u_j(\mathbf{y}, t) \nabla^2 U_i(\mathbf{x}, t)}. \end{aligned} \quad (5.83)$$

Using the definition of $\mathcal{R}_{ij}$ given in Eq. (2.30) it follows that the first two terms on the left-hand side of Eq. (5.83) may be written as

$$\overline{\rho u_i(\mathbf{x}, t) \frac{\partial U_j}{\partial t}(\mathbf{y}, t) + \rho u_j(\mathbf{y}, t) \frac{\partial U_i}{\partial t}(\mathbf{x}, t)} = \rho \frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{x}, \mathbf{y}, t) \quad (5.84)$$

since terms such as $\overline{u_i(\mathbf{x}, t) \partial \overline{U_j}(\mathbf{y}, t) / \partial t} \equiv 0$. The next two terms in Eq. (5.83), coming from the advection term, give

$$\begin{aligned} & \overline{\rho u_i(\mathbf{x}, t) U_k(\mathbf{y}, t) \frac{\partial U_j}{\partial y_k}(\mathbf{y}, t) + \rho u_j(\mathbf{y}, t) U_k(\mathbf{x}, t) \frac{\partial U_i}{\partial x_k}(\mathbf{x}, t)} = \\ & \overline{\rho u_i(\mathbf{x}, t) u_k(\mathbf{y}, t) \frac{\partial \overline{U_j}}{\partial y_k}(\mathbf{y}, t) + \rho u_j(\mathbf{y}, t) u_k(\mathbf{x}, t) \frac{\partial \overline{U_i}}{\partial x_k}(\mathbf{x}, t)} + \\ & \overline{\rho \overline{U_k}(\mathbf{y}, t) u_i(\mathbf{x}, t) \frac{\partial u_j}{\partial y_k}(\mathbf{y}, t) + \rho \overline{U_k}(\mathbf{x}, t) u_j(\mathbf{y}, t) \frac{\partial u_i}{\partial x_k}(\mathbf{x}, t)} + \\ & \overline{\rho u_i(\mathbf{x}, t) u_k(\mathbf{y}, t) \frac{\partial u_j}{\partial y_k}(\mathbf{y}, t) + \rho u_j(\mathbf{y}, t) u_k(\mathbf{x}, t) \frac{\partial u_i}{\partial x_k}(\mathbf{x}, t)}. \end{aligned} \quad (5.85)$$

The first two terms on the right-hand side of Eq. (5.85) are equal to

$$\rho \mathcal{R}_{ik}(\mathbf{x}, \mathbf{y}, t) \frac{\partial \overline{U_j}}{\partial y_k}(\mathbf{y}, t) + \rho \mathcal{R}_{jk}(\mathbf{y}, \mathbf{x}, t) \frac{\partial \overline{U_i}}{\partial x_k}(\mathbf{x}, t). \quad (5.86)$$

Furthermore, differentiation of Eq. (2.30) gives

$$\frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x}, \mathbf{y}, t) = u_i(\mathbf{x}, t) \frac{\partial u_j}{\partial y_k}(\mathbf{y}, t) \quad (5.87)$$

and similarly for x_k derivatives, so that the third and fourth terms on the right-hand side of Eq. (5.85) take the form of convection terms

$$\rho \bar{U}_k(\mathbf{y}, t) \frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x}, \mathbf{y}, t) + \rho \bar{U}_k(\mathbf{x}, t) \frac{\partial \mathcal{R}_{ij}}{\partial x_k}(\mathbf{x}, \mathbf{y}, t). \quad (5.88)$$

As far as the last two terms on the right-hand side of Eq. (5.85) are concerned, they may be written using Eq. (2.31) as

$$\rho u_i(\mathbf{x}, t) u_k(\mathbf{y}, t) \frac{\partial u_j}{\partial y_k}(\mathbf{y}, t) = \rho \frac{\partial S_{jk,i}}{\partial y_k}(\mathbf{y}, \mathbf{x}, t) \quad (5.89)$$

and

$$\rho u_j(\mathbf{y}, t) u_k(\mathbf{x}, t) \frac{\partial u_i}{\partial x_k}(\mathbf{x}, t) = \rho \frac{\partial S_{ik,j}}{\partial x_k}(\mathbf{x}, \mathbf{y}, t) \quad (5.90)$$

where the fact that

$$u_i(\mathbf{x}, t) \frac{\partial u_j}{\partial x_j}(\mathbf{x}, t) u_k(\mathbf{y}, t) = 0 \quad (5.91)$$

has been used as implied by incompressibility.

To treat the contribution to Eq. (5.83) from the terms containing pressure, introduce the two-point pressure-velocity correlation vector

$$\mathcal{K}_i(\mathbf{x}, \mathbf{y}, t) = \overline{u_i(\mathbf{x}, t) p(\mathbf{y}, t)} \quad (5.92)$$

and see that

$$u_i(\mathbf{x}, t) \frac{\partial p}{\partial y_j}(\mathbf{y}, t) + u_j(\mathbf{y}, t) \frac{\partial p}{\partial x_i}(\mathbf{x}, t) = \frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x}, \mathbf{y}, t) + \frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y}, \mathbf{x}, t). \quad (5.93)$$

Finally, the viscous terms in Eq. (5.83) become

$$\overline{\mu u_i(\mathbf{x}, t) \nabla^2 U_j(\mathbf{y}, t)} + \overline{\mu u_j(\mathbf{y}, t) \nabla^2 U_i(\mathbf{x}, t)} = \mu \frac{\partial^2 \mathcal{R}_{ij}}{\partial y_k^2}(\mathbf{x}, \mathbf{y}, t) + \mu \frac{\partial^2 \mathcal{R}_{ij}}{\partial x_k^2}(\mathbf{x}, \mathbf{y}, t). \quad (5.94)$$

Putting the above results together it has been shown that Eq. (5.83) becomes

$$\begin{aligned} & \frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{x}, \mathbf{y}, t) + \bar{U}_k(\mathbf{y}, t) \frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x}, \mathbf{y}, t) + \bar{U}_k(\mathbf{x}, t) \frac{\partial \mathcal{R}_{ij}}{\partial x_k}(\mathbf{x}, \mathbf{y}, t) \\ &= -\mathcal{R}_{ik}(\mathbf{x}, \mathbf{y}, t) \frac{\partial \bar{U}_j}{\partial y_k}(\mathbf{y}, t) - \mathcal{R}_{jk}(\mathbf{y}, \mathbf{x}, t) \frac{\partial \bar{U}_i}{\partial x_k}(\mathbf{x}, t) \\ & \quad - \frac{\partial S_{jk,i}}{\partial y_k}(\mathbf{y}, \mathbf{x}, t) - \frac{\partial S_{ik,j}}{\partial x_k}(\mathbf{x}, \mathbf{y}, t) - \frac{1}{\rho} \frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x}, \mathbf{y}, t) \\ & \quad - \frac{1}{\rho} \frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y}, \mathbf{x}, t) + \nu \frac{\partial^2 \mathcal{R}_{ij}}{\partial y_k^2}(\mathbf{x}, \mathbf{y}, t) + \nu \frac{\partial^2 \mathcal{R}_{ij}}{\partial x_k^2}(\mathbf{x}, \mathbf{y}, t). \end{aligned} \quad (5.95)$$

When $\mathbf{x} = \mathbf{y}$, $\mathcal{R}_{ij}(\mathbf{x}, \mathbf{x}, t) = R_{ij}(\mathbf{x}, t)$, and it may be shown that Eq. (5.95) becomes identical to Eq. (3.53). This connection suggests that the first two terms on the right-hand side of Eq. (5.95) are "production" terms. The remaining terms acquire meaning by noting their similarity to the corresponding terms in Eq. (3.53).

The formidable complexity of Eq. (5.95) can be reduced somewhat by applying the relation to the specific case of homogeneous, isotropic turbulence. Since $\overline{U}_k(\mathbf{y}, t) = \overline{U}_k(\mathbf{x}, t)$ in homogeneous turbulence, and using results like Eq. (5.87), it follows that the two convection terms on the left-hand side of Eq. (5.95) sum to zero. Uniformity of $\overline{U}_i$ also implies that the two production terms on the right-hand side of Eq. (5.95) are zero.

The simplification for homogeneous turbulence used in Eq. (4.1) can be generalized to include the statements that

$$S_{ij,k}(\mathbf{x}, \mathbf{y}, t) = S_{ij,k}(\mathbf{y} - \mathbf{x}, t) \quad (5.96)$$

and

$$\mathcal{K}_i(\mathbf{x}, \mathbf{y}, t) = \mathcal{K}_i(\mathbf{y} - \mathbf{x}, t), \quad (5.97)$$

where for convenience the same symbols $\mathcal{R}_{ij}$, $S_{ij,k}$, and $\mathcal{K}_i$ on the right-hand side are adopted; their applicability to homogeneous turbulence is implied by the appearance of one less argument than their more general counterparts. Using these relations, it follows that

$$\frac{\partial S_{jk,i}}{\partial y_k}(\mathbf{y}, \mathbf{x}, t) = -\frac{\partial S_{jk,i}}{\partial r_k}(\mathbf{x} - \mathbf{y}, t) \quad (5.98)$$

and

$$\frac{\partial S_{ik,j}}{\partial x_k}(\mathbf{x}, \mathbf{y}, t) = -\frac{\partial S_{ik,j}}{\partial r_k}(\mathbf{y} - \mathbf{x}, t), \quad (5.99)$$

and that

$$\frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x}, \mathbf{y}, t) = \frac{\partial \mathcal{K}_i}{\partial r_j}(\mathbf{y} - \mathbf{x}, t) \quad (5.100)$$

and

$$\frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y}, \mathbf{x}, t) = \frac{\partial \mathcal{K}_j}{\partial r_i}(\mathbf{x} - \mathbf{y}, t). \quad (5.101)$$

Putting together the various results, it is found that the two-point velocity correlation tensor in homogeneous turbulence is governed by the equation

$$\begin{aligned} \frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{r}, t) &= \frac{\partial S_{jk,i}}{\partial r_k}(-\mathbf{r}, t) + \frac{\partial S_{ik,j}}{\partial r_k}(\mathbf{r}, t) \\ &\quad - \frac{1}{\rho} \frac{\partial \mathcal{K}_i}{\partial r_j}(\mathbf{r}, t) - \frac{1}{\rho} \frac{\partial \mathcal{K}_j}{\partial r_i}(-\mathbf{r}, t) + 2\nu \frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k^2}(\mathbf{r}, t). \end{aligned} \quad (5.102)$$

Contracting the indices in Eqs. (5.100) and (5.101), noting the definition of $\mathcal{K}_i$ in Eq. (5.92) and using the incompressibility condition gives in both cases

$$\frac{\partial \mathcal{K}_i}{\partial r_i}(\mathbf{r}, t) = 0. \quad (5.103)$$

Now taking a trace of Eq. (5.102) and using (5.103) gives

$$\frac{\partial \mathcal{R}_{ii}}{\partial t}(\mathbf{r}, t) = \frac{\partial S_{ik,i}}{\partial r_k}(-\mathbf{r}, t) + \frac{\partial S_{ik,i}}{\partial r_k}(\mathbf{r}, t) + 2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\mathbf{r}, t), \quad (5.104)$$

which shows that the time rate of change of the trace of the two-point velocity correlation tensor depends on a balance between viscous diffusion, given in the last term, and the two terms depending on the two-point triple velocity correlation tensor. The latter represent the process by which vortex stretching brings energy to small dissipative scales.

See Hise

Section 1-5

5.6 Self-Preservation of the Karman-Howarth Equation

Using $R_{ij} = f(f)$ & $S_{ij} = f(k)$ the $\frac{\partial R_{ij}}{\partial t}(v, t)$ equation becomes

$$\frac{\partial (\bar{u}^2 f)}{\partial t} = \bar{u}^2 \frac{3}{2} \left(k_v + \frac{4}{3} k \right) + 2v \bar{u}^2 \left(f_{vv} + \frac{4}{v} f_v \right) \quad \text{Pope pp. 201-202} \quad \text{Hence 3-3}$$

A2 + A3

which relates $f(v, t)$, $k(v, t)$ & $\bar{u}_{rms}(t)$. Using

$$\text{TS for } f \text{ & } k \text{ & also } \varepsilon = \frac{30 \bar{u}^3}{\lambda_f^2} = \frac{15 v \bar{u}^2}{\lambda_f^2}$$

equations $\frac{dk}{dt} = -\varepsilon$ &

$$\frac{d\varepsilon}{dt} = S_K R_T^{1/2} \varepsilon^2 / k - C^* \varepsilon^2 / k \text{ can be derived}$$

by gathering terms depending on some power of v .

$$\text{Assuming self similarity } f(v, t) = \tilde{f}(v/L(t)) \\ k(v, t) = \tilde{k}(v/L(t))$$

$$2\gamma^{-4} \frac{d}{d\gamma} \left(\gamma^4 \frac{d\tilde{f}}{d\gamma} \right) + \gamma \frac{d\tilde{f}}{d\gamma} \left(\frac{7}{3} 60 - 5 \right) + 10 \tilde{f} =$$

$$R_\lambda \left(\frac{7}{6} S_{K0} \gamma \frac{d\tilde{f}}{d\gamma} - \gamma^{-4} \frac{d}{d\gamma} \left(\gamma^4 \tilde{k} \right) \right) \quad v = v/\lambda_f$$

Single ODE for $\tilde{f}(\gamma)$ & $\tilde{k}(\gamma)$ with $R_\lambda(t)$ parameter

For self similarity both RHS & LHS = 0
for $R_\lambda \neq 0$ otherwise LHS multivalued
for $R_\lambda(t)$

LHS = confluent hypergeometric equation
with solution

$$\tilde{f}(\eta) = M\left(\frac{1}{6_0^* - 1}, \frac{\xi}{2}, -\frac{S(6_0^* - 1)}{4} \eta^2\right) \quad M = \text{confluent hypergeometric function}$$

Integration RHS

$$\tilde{g}(\eta) = \frac{7}{6} S K_0 \frac{1}{\eta^4} \int_0^\eta S^5 \frac{d\tilde{f}}{dS} dS$$

with $\tilde{f}(\eta)$ from LHS solution. Thus self similarity provides $\tilde{f}$ & $\tilde{g}$ for chosen 6_0 & S_{K_0} .

Recall complete similarity not possible in isotopic decay so $\tilde{f}$ & $\tilde{g}$ must be $f(t)$.
However still useful examine high & low R_T equilibrium solutions

(1) small R_T near $R_{T\infty} = 0$: RHS small & wry $6_0^* = 715$ for first period

$$\tilde{f}(\eta) = M\left(\frac{\xi}{2}, \frac{\xi}{2}, -\eta^2/2\right) = e^{-\eta^2/2} \quad \text{in agreement} \quad f(v, t) = e^{-v^2/2\tau_0^2}$$

$$(2) R_T = 0, \quad \tilde{g}(\eta) = \frac{7}{6} S K_0 \frac{1}{\eta^4} [(3\xi^5 + 5\xi^3 + 15\xi)e^{-\eta^2/2}$$

$$\text{erf}(\eta) = \frac{2}{\pi} \int_0^\eta e^{-S^2} dS \quad -15\sqrt{\frac{\pi}{2}} \text{erf}\left(\frac{\eta}{\sqrt{2}}\right)]$$

$\tilde{k}$ decays slower
for large η than
Gaussian $\tilde{f}$

$$\tilde{k}(\eta) \sim \eta^{-4} \text{ as } \eta \rightarrow \infty$$

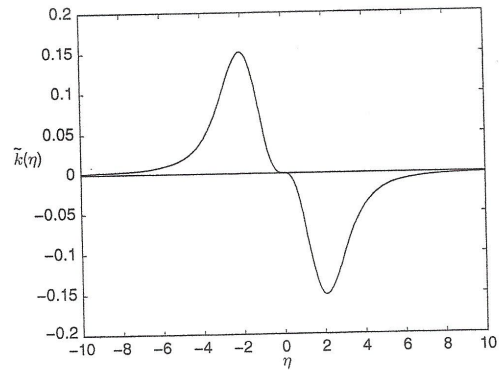


Figure 5.7 $\tilde{k}$ in the final period.

(3) For high Re equilibrium

$$R_\chi = \text{constant} \neq 0$$

1. $\tilde{f}, \tilde{k}$ equation remains single equation
with two unknown functions. Thus, unlike
self similar $\chi < 1$ equations Kurman-Howarth
equation not solvable without additional
assumptions.

Additional Discussion Kurman-Howarth Equation:

(1) closure problem, i.e., one equation two unknowns

(2) k and v terms represent magnetic viscosity
processes, respectively.

(3) at $v=0$ k term = 0 since $k(r, t) = k'''(r^3/3) + k^v(r^5/5), \dots$
(continuity $\Rightarrow k'(0, t) = 0$), whereas

$$\left[\frac{1}{v} \frac{\partial}{\partial r} \left(r^4 \frac{\partial f}{\partial r} \right) \right]_{r=0} = 5 f''(0, t) = -5 / \lambda_g(t)^2$$

$$\text{i.e.} \quad \frac{d}{dt} \overline{u^2} = -10v \frac{\overline{u^2}}{\lambda_g^2} = -\frac{2}{3} \varepsilon \quad \lambda_g^2 = \frac{15v \overline{u^2}}{\varepsilon}$$

(4) Energy cascade for high Re hypothesis is that the energy transfer from large to smaller scales is an inertial process for $r \gg \eta$; consequently, ϵ term is responsible for this process

(5) If u was Gaussian then $\epsilon(r, t)$ would be zero; hence, energy cascade depends on non Gaussian aspects of velocity field

Skewness of velocity derivative

$$\overline{u^3} \epsilon'''(0, t) = \overline{\left(\frac{\partial u_i}{\partial x_i}\right)^3} = S \left(\frac{\epsilon}{15\nu} \right)^{3/2} = -\frac{2}{35} \overline{w_i w_i} \frac{\partial u_i}{\partial x_i}$$

$$S = \overline{u_{i,1}^3} / (\overline{u_{i,1}^2})^{3/2} < 0$$

∴ connection between S & vortex stretching & transfer of energy between different scales

One Kolomogorov 4/5 law

The Karman-Howarth equation can be re-expressed in terms of structure functions $D_{LL}(r, t)$ as

$$D_{LLL}(r, t) = [\overline{u_1(x + \hat{e}, r, t) - u_1(x, t)}]^3$$

$$\frac{\partial}{\partial z} D_{LLL} + \frac{1}{3v^4} \frac{\partial}{\partial v} (v^4 D_{LLL}) = \frac{2v}{v^4} \frac{\partial}{\partial v} \left(v^4 \frac{\partial D_{LL}}{\partial v} \right) - \frac{4}{3} \varepsilon$$

$$\frac{3}{15} \int_0^v s^4 \frac{\partial}{\partial s} D_{LL}(s, t) ds = 6v \frac{\partial D_{LL}}{\partial v} - D_{LLL} - \frac{4}{3} \varepsilon v$$

For isotropic turbulence in the inertial subrange

inertial term = 0 $D_{LLL}(v, t) = -\frac{4}{3} \varepsilon v$

✓ term negligible

$$S' = D_{LLL}(v, t) / D_{LL}(v, t)^{3/2}$$

= constant

$$\Rightarrow D_{LL}(v, t) = \left(\frac{-4}{5S'} \right)^{2/3} (\varepsilon v)^{2/3}$$

Same as Kolomogorov hypothesis, which shows consistency between it and WS equations; and relates Kolomogorov constant to shear stress S'

5.7 Energy Spectrum Equation (Huze Section 3-5)

The equation for $R_{ij}(\underline{r}, t)$ was derived by taking the average of $u_i(\underline{x}, t)$ times u_j at $\underline{y}$ + average of $u_j(\underline{y}, t)$ times u_i at $\underline{x}$.

3D Fourier transform of $R_{ij}(\underline{r}, t)$

$$\underline{Z}_{ij}(\underline{k}, t) = (2\pi)^{-3} \int e^{i\underline{k} \cdot \underline{r}} R_{ij}(\underline{r}, t) d\underline{r}$$

$$R_{ij}(\underline{r}, t) = \int e^{-i\underline{k} \cdot \underline{r}} \underline{Z}_{ij}(\underline{k}, t) d\underline{k}$$

$$d\underline{k} = dk_1 dk_2 dk_3, \quad R_{ij} \text{ and } \underline{Z}_{ij} = f(\underline{x})$$

decomposes R_{ij} = turbulent velocity correlations into contributions from continuous range of scales as represented by $e^{i\underline{k} \cdot \underline{r}}$

Fourier transform of the R_{ij} equation:

$$\frac{\partial \underline{Z}_{ij}(\underline{k}, t)}{\partial t} = T_{ij}(\underline{k}, t) + P_{ij}(\underline{k}, t) - 2k^2 \nu \underline{Z}_{ij}(\underline{k}, t)$$

$$T_{ij}(\underline{k}, t) = (2\pi)^{-3} \int e^{i\underline{k} \cdot \underline{r}} S_{ij}(\underline{r}, t) d\underline{r}$$

$$S_{ij}(\underline{r}, t) = \frac{\partial S_{ij1}}{\partial r_1}(-\underline{r}, t) + \frac{\partial S_{ij1}}{\partial r_2}(\underline{r}, t)$$

T_{ij} = rate of transfer of energy (gain or loss) between different scales of turbulent motion due to vortex stretching & reorientation

$$P_{ij}(\underline{k}, \underline{z}) = - (2\pi)^{-3} \int_V e^{i\underline{k} \cdot \underline{z}} \left[\frac{1}{\rho} \frac{\partial K_i}{\partial x_j}(\underline{r}, t) + \frac{1}{\rho} \frac{\partial K_j}{\partial x_i}(\underline{r}, t) \right] dV$$

$$K_i(\underline{x}, \underline{y}, t) = \overline{u_i(\underline{x}, t) p(\underline{y}, t)} \quad \text{two-point velocity pressure correlation}$$

$P_{ij} = 0$ for isotropic turbulence
 = influence of pressure field on buoyancy
 anisotropic turbulence isotropic

$$-2\nu k^2 \underline{\underline{z}}_{ij} = \text{viscous dissipation term}$$

Contribution of indices P_{ij} drops out of
 integrating over spherical shell converts
 $\underline{\underline{z}}_{ij}$ to $E = \frac{1}{2} \int_{|\underline{k}|=k} \underline{\underline{z}}_{ij}(\underline{k}, t) d\Omega$, $K = \int_0^\infty E(k, t) 4\pi k^2 dk$

$$\frac{\partial E(k, t)}{\partial t} = T(k, t) - 2\nu k^2 E(k, t)$$

$$T(k, t) = \frac{1}{2} \int_{|\underline{k}|=k} T_{ii}(\underline{k}, t) d\Omega = \text{transfer term}$$

$T_{ii} = \text{Fourier transform}$

$$S_{ii}(\underline{r}, t) = \overline{u_{rms}^3} \frac{1}{r^2} \frac{d}{dr} \left(r^3 \frac{d\lambda}{dr}(\underline{r}, t) + 4\nu r^2 \lambda(\underline{r}, t) \right)$$

using $S_{ij}(\underline{r}) = f(k(r))$ equation. Note here
 $k(r) = k(r, t) = S_{11}(r\hat{e}_1) / \overline{u_{rms}^3}$ correlation function
 not to be confused with wave number k .

That is $S_{ii} = f(r)$ only in isotropic turbulence

using $S_{ii}(r, t)$ into $T_{ij}(k, t)$ equation

$$T_{ii}(k, t) = \frac{1}{2\pi^2} \int_0^\infty S_{ii}(r, t) r^2 \frac{\sin kr}{kr} dr = f(k) \neq f(k)$$

Substituting $S_{ii}(r, t)$ & integrating by parts twice $\left[\int u dv = uv - \int v du = uv - \int u dv \right]$
or $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

& substitution into $T(k, t) = \frac{1}{2} \int_{-\infty}^\infty T_{ii}(k, t) dk$

$$T(k, t) = \frac{u_{rms}^3}{\pi} \int_0^\infty r [(3 - k^2 r^2) \sin kr - 3kr \cos kr] \Delta(r, t) dr$$

Which shows that $\Delta(r, t)$ determines the rate of energy transfer between the scales of the turbulence

$$\left. \begin{array}{l} \int_{k=0}^\infty T(k, t) dk = 0 \\ \text{ie net energy transfer between scales} = 0 \text{ or in other words gains \& losses are conserved!} \end{array} \right\} \begin{array}{l} \text{integrating} \\ \frac{\partial E}{\partial t} \text{ eq.} \\ \text{at using} \end{array} \left\{ \begin{array}{l} K(t) = \int_0^\infty E(k, t) dk \\ \Sigma = 2\nu \int_0^\infty k^2 E(k, t) dk \\ \frac{dK}{dt} = -\Sigma \end{array} \right.$$

When $f(r, t)$ is known $E(k, t)$ can be obtained from $E(k) = \frac{u^2}{\pi} \int_0^\infty (3f(r) + r f'(r)) \Delta r \sin kr dr$

& $T(k, t)$ can be obtained from $\frac{\partial E}{\partial t} = T - 2\nu k^2 E$ without need for evaluating $T(k, t)$ from above equation

An important example is for the final period of decay of isotropic turbulence where

$$f(r, z) = e^{-r^2/2\lambda_g^2} \quad \text{ie Gaussian function}$$

$$\text{Then } E(k, t) = \frac{u_{rms}^2 \lambda_g}{\sqrt{2\pi}} (k\lambda_g)^4 e^{-\frac{1}{2}(k\lambda_g)^2}$$

$$\text{which is consistent with } E(k) = -4\pi k^4 E_1(k)$$

Substituting into $\frac{\partial E}{\partial t}$ equation gives

$$T(k, t) = E(k, t) \frac{u_{rms}}{\lambda_g} ((k\lambda_g)^2 - 5)$$

$k < \sqrt{5}/\lambda_g$ lose energy
to those for $k > \sqrt{5}/\lambda_g$

$$k_2 = 2/\lambda_g \quad \text{peak } E$$

$$k_1 = \sqrt{6}/\lambda_g \quad \text{peak dissipation}$$

For this example
the k had not
well separated
due to low Re

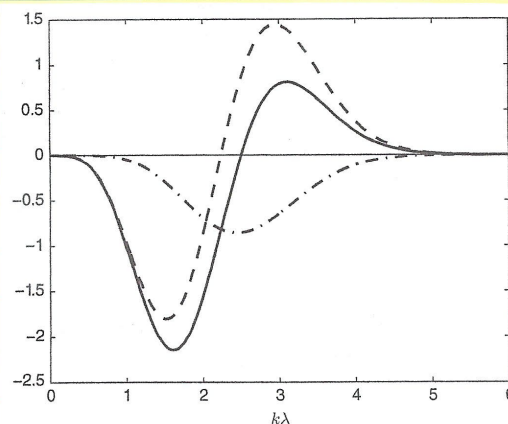


Figure 5.8 Energy spectrum budget in final period. —, $\partial E/\partial t$; ---, transfer term; - · -, dissipation. In this illustration $Re_\lambda = 10$, with λ denoting λ_g .

In particular, no marked range for above form $E(k, t)$ equation. As λ_g rises (Fig 5.5) the balance in Fig 5.8 shifts to small wave numbers ie larger scales (ie higher wave numbers lose all their energy before lower wave numbers)

5.8 Energy Spectrum Equation via Fourier Analysis of Velocity Field

Here transfer physics analyzed based on Fourier Analysis WS, whereas previous approach used R1 equation leading to analysis based on $u(x, z)$, which shows that energy transfer occurs due to interactions between scales at specific combinations of wave numbers

5.8.1 Fourier Analysis on a Cubic Region

$$u_i(x, z) = \sum_{\underline{l}} \hat{u}_i(\underline{l}, z) e^{i\underline{l} \cdot \underline{x}} \quad \underline{l} = 2\pi \underline{n} / L$$

Fourier transform WS $\underline{n} = (n_1, n_2, n_3)$
 straight forward except $-\infty$ to ∞
 nonlinear convection/advection term

$$\hat{u}_i \frac{\partial u_i}{\partial x_j}(\underline{l}, z) = \frac{1}{L^3} \int_V u_i(x, z) \frac{\partial u_i}{\partial x_j}(x, z) e^{-i\underline{l} \cdot \underline{x}} d\underline{x}$$

$$\text{note } \hat{u}_i(\underline{l}, z) = \frac{1}{L^3} \int_V u_i(x, z) e^{-i\underline{l} \cdot \underline{x}} d\underline{x} \quad \text{inverse transform}$$

$$\hookrightarrow \frac{\partial u_i}{\partial x_j}(x, z) = i \sum_{\underline{l}} l_j \hat{u}_i(\underline{l}, z) e^{i\underline{l} \cdot \underline{x}}$$

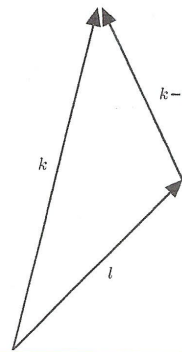
$$\text{Such that } \hat{u}_i \frac{\partial u_i}{\partial x_j}(\underline{l}, z) = i \sum_{\underline{l}} \sum_{\underline{m}} l_j \hat{u}_i(\underline{l}, z) \hat{u}_j(\underline{m}, z)$$

$$\begin{aligned} \text{using } \langle e^{i\underline{l} \cdot \underline{x}}, e^{i\underline{l}' \cdot \underline{x}} \rangle &= \frac{1}{L^3} \int_V d\underline{x} e^{-i(\underline{l} - \underline{l}') \cdot \underline{x}} \\ &= \int_V e^{i(\underline{l} - \underline{l}') \cdot \underline{x}} d\underline{x} = 0 \quad \underline{l} \neq \underline{l}' \quad \text{due to orthogonality} \end{aligned}$$

$$\text{i.e. } u_j \frac{\partial u_i}{\partial x_j}(\underline{k}, t) = i \sum_{\underline{l}} l_j \hat{u}_i(\underline{l}, t) \hat{u}_j(\underline{k} - \underline{l}, t)$$

Shows contributions convection/advection term at $\underline{k}$ only come from triads of wave numbers, i.e., influence of $\underline{l}$ on $\underline{k}$ includes $\underline{k} - \underline{l}$

Figure 5.9 Triads of wave numbers.



Analyzing all terms in the NS equation

$$\begin{aligned} \frac{\partial \hat{u}_i}{\partial t}(\underline{k}, t) = & -i \sum_{\underline{l}} l_j \hat{u}_i(\underline{l}, t) \hat{u}_j(\underline{k} - \underline{l}, t) \\ & + i k_i \frac{\hat{p}(\underline{k}, t)}{k} - \nu k^2 \hat{u}_i(\underline{k}, t) \end{aligned}$$

$$\text{where } p(\underline{x}, t) = \sum_{\underline{k}} \hat{p}(\underline{k}, t) e^{i \underline{k} \cdot \underline{x}}$$

$$\hat{p}(\underline{k}, t) = \frac{1}{L^3} \int_V p(\underline{x}, t) e^{-i \underline{k} \cdot \underline{x}} d\underline{x}$$

Fourier transform continuity equation $\frac{\partial u_i}{\partial x_i} = 0$ yields

$k_i \hat{u}_i(\underline{k}, t) = 0$ which shows that $\hat{u}_i(\underline{k}, t)$ is $\perp$ to $\underline{k}$ i.e. for any $\underline{k}$, $\hat{u}_i(\underline{k}, t)$ is tangent to surface of sphere of radius $|\underline{k}|$ centered at origin

$\underline{k} \cdot$ Fourier transform WS and using continuity yields

$$\hat{p}(\underline{k}, t) = -\frac{k_i}{k^2} \sum_j k_j \hat{u}_j(\underline{k}, t) \hat{u}_j(\underline{k} - \underline{k}, t)$$

$$\text{since } k_j \hat{u}_j(\underline{k} - \underline{k}, t) = k_j \hat{u}_j(\underline{k} - \underline{k}, t)$$

using this in Fourier transform WS

$$\frac{\partial \hat{u}_i}{\partial t}(\underline{k}, t) + v k_i \hat{u}_i(\underline{k}, t) = -i P_{ij}(\underline{k}) k_m \sum_n \hat{u}_j(\underline{k}, t) \hat{u}_m(\underline{k} - \underline{k}, t)$$

$$P_{ij}(\underline{k}) = \delta_{ij} - \frac{k_i k_j}{k^2} = \text{projection operator}$$

maps any vector onto

for any vector $\underline{v}$ its component $\perp$ to $\underline{k}$

$$\text{for example } P_{ij}(\underline{k}) v_j = v_i - (v_j k_j) k_i / k^2$$

$$\text{is } \perp \text{ to } k_i \text{ since } (P_{ij}(\underline{k}) v_j) k_i = 0$$

$P_{ij}(\underline{k})$ term consistent LHS which is also $\perp$ to $\underline{k}$

Equivalent form $RHS = M_{ijm}(\underline{k}) \sum_{\underline{l}} \hat{u}_i(\underline{l}, t) \hat{u}_m(\underline{l}-\underline{l}, t)$

$$M_{ijm} = -\frac{i}{2} (\underline{k}_m P_{ij}(\underline{k}) + \underline{k}_j P_{im}(\underline{k}))$$

Dynamics equation for discrete energy spectrum component $E_{\underline{k}} = |\hat{u}(\underline{k}, t)|^2 / 2$ is obtained by taking the sum of the Fourier transformed NS times $\hat{u}_i^*(\underline{k}, t)$ and the complex conjugate of the Fourier transform of the NS times $\hat{u}_i(\underline{k}, t)$

$$\frac{\partial E_{\underline{k}}}{\partial t}(\underline{k}) + 2 \gamma \underline{k}^2 E_{\underline{k}}(\underline{k}) =$$

$$\frac{1}{2} M_{ijm}(\underline{k}) \sum_{\underline{l}} (\hat{u}_i(-\underline{l}) \hat{u}_j(\underline{l}) \hat{u}_m(\underline{k}-\underline{l}) - (\hat{u}_i(\underline{k}) \hat{u}_j(\underline{l}) \hat{u}_m(-\underline{k}-\underline{l})))$$

$\underline{l}$ replaced by $-\underline{l}$ for convenience

RHS accounts for energy transfer between wave numbers showing the triadic nature of such exchanges

A dynamical equation for the discrete energy spectrum component $E_k = |\hat{u}(\mathbf{k}, t)|^2/2$ that was introduced previously in Section 2.2.4 may be derived by taking the average of the sum of Eq. (5.136) times $\hat{u}_i^*(\mathbf{k}, t)$ and the complex conjugate of Eq. (5.136) times $\hat{u}_i(\mathbf{k}, t)$. The result is

$$\frac{\partial E_k}{\partial t}(t) + 2\nu k^2 E_k(t) = \frac{1}{2} M_{ijm}(\mathbf{k}) \sum_l (\hat{u}_i(-\mathbf{k}) \hat{u}_j(l) \hat{u}_m(\mathbf{k}-l) - \hat{u}_i(\mathbf{k}) \hat{u}_j(l) \hat{u}_m(-\mathbf{k}-l)), \quad (5.140)$$

where l has been replaced with $-l$ in the second term for later convenience. The term on right-hand side of Eq. (5.140) accounts for the energy transfer between wave numbers. The triadic nature of such exchanges is evident in these expressions.

5.8.2 Limit of Infinite Space

To acquire an alternative form of the dynamical equation for $E(k, t)$ consider the limiting form of Eq. (5.140) as $L \rightarrow \infty$. With the goal of taking this limit define

$$E_{ij}^L(\mathbf{k}, t) = \left(\frac{L}{2\pi}\right)^3 \overline{\hat{u}_i(\mathbf{k}) \hat{u}_j(-\mathbf{k})}, \quad (5.141)$$

which becomes, after using Eqs. (2.57) and (2.62),

$$E_{ij}^L(\mathbf{k}, t) = \left(\frac{1}{2\pi}\right)^3 \int_{V_L} \mathcal{R}_{ij}(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}. \quad (5.142)$$

In the limit as $L \rightarrow \infty$ the right-hand side of Eq. (5.142) becomes the Fourier transform of $\mathcal{R}_{ij}(\mathbf{r}, t)$ so it has been established that

$$\lim_{L \rightarrow \infty} E_{ij}^L(\mathbf{k}, t) = \mathcal{E}_{ij}(\mathbf{k}, t). \quad (5.143)$$

In this limit the values of $\mathbf{k}$ become closer and closer together, and at the end of the limiting process $\mathbf{k}$ covers all real vectors and not just the discrete set where $E_{ij}^L(\mathbf{k}, t)$ is defined at a particular value of L .

This same argument needs to be repeated in the case of triple velocity correlations. Thus, define

$$T_{ijn}^L(\mathbf{k}, l, t) = \left(\frac{L}{2\pi}\right)^6 \overline{\hat{u}_i(\mathbf{k}) \hat{u}_j(l) \hat{u}_n(-\mathbf{k}-l)} \quad (5.144)$$

where, by a generalization of the argument leading to Eq. (2.61), use has been made of the fact that

$$\overline{\hat{u}_i(\mathbf{k}, t) \hat{u}_j(l, t) \hat{u}_n(\mathbf{m}, t)} = 0 \quad \text{unless} \quad \mathbf{k} + l + \mathbf{m} = 0. \quad (5.145)$$

Substituting for the Fourier components using Eq. (2.52) then gives

$$T_{ijn}^L(\mathbf{k}, l, t) = \left(\frac{L}{2\pi}\right)^6 \frac{1}{L^9} \int_{V_L} \int_{V_L} \int_{V_L} \overline{u_i(\mathbf{x}, t) u_j(\mathbf{y}, t) u_n(\mathbf{z}, t)} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{z}) - i\mathbf{l}\cdot(\mathbf{y}-\mathbf{z})} d\mathbf{x} d\mathbf{y} d\mathbf{z}. \quad (5.146)$$

In homogeneous turbulence the triple velocity correlation within the integral depends on just the two vectors $\mathbf{r} = \mathbf{x} - \mathbf{z}$ and $\mathbf{s} = \mathbf{y} - \mathbf{z}$. This justifies the definition

$$S_{ijn}(\mathbf{r}, \mathbf{s}, t) \equiv \overline{u_i(\mathbf{z} + \mathbf{r}, t) u_j(\mathbf{z} + \mathbf{s}, t) u_n(\mathbf{z}, t)}, \quad (5.147)$$

which is to be contrasted with the two-point triple correlation S_{ijn} in Eq. (2.31). Changing the $\mathbf{x}$ and $\mathbf{y}$ variables in Eq. (5.146) to $\mathbf{r}$ and $\mathbf{s}$, respectively, using Eq. (5.147), and carrying out the $\mathbf{z}$ integration in Eq. (5.146) yields

$$T_{ijn}^L(\mathbf{k}, \mathbf{l}, t) = \left(\frac{1}{2\pi}\right)^6 \int_{V_L} \int_{V_L} S_{ijn}(\mathbf{r}, \mathbf{s}, t) e^{-i\mathbf{k}\cdot\mathbf{r} - i\mathbf{l}\cdot\mathbf{s}} d\mathbf{r} d\mathbf{s}. \quad (5.148)$$

In the limit as $L \rightarrow \infty$ this becomes

$$T_{ijn}(\mathbf{k}, \mathbf{l}, t) = \left(\frac{1}{2\pi}\right)^6 \int \int S_{ijn}(\mathbf{r}, \mathbf{s}, t) e^{-i\mathbf{k}\cdot\mathbf{r} - i\mathbf{l}\cdot\mathbf{s}} d\mathbf{r} d\mathbf{s} \quad (5.149)$$

as the Fourier transform of $S_{ijn}(\mathbf{r}, \mathbf{s}, t)$.

Now we are prepared to consider the limit of Eq. (5.140) as $L \rightarrow \infty$. Multiply this equation by $(L/2\pi)^3$ and note through Eqs. (5.141), (5.143), and (4.66) that

$$\lim_{L \rightarrow \infty} \left(\frac{L}{2\pi}\right)^3 E_k = \frac{1}{2} \mathcal{E}_k(\mathbf{k}, t) = \frac{E(k, t)}{4\pi k^2}. \quad (5.150)$$

Moreover, it may be calculated using Eq. (5.144) that

$$\begin{aligned} \lim_{L \rightarrow \infty} \left(\frac{L}{2\pi}\right)^3 \sum_{\mathbf{l}} \overline{\hat{u}_j(\mathbf{l}) \hat{u}_m(\mathbf{k} - \mathbf{l}) \hat{u}_i(-\mathbf{k})} &= \lim_{L \rightarrow \infty} \sum_{\mathbf{l}} \left(\frac{2\pi}{L}\right)^3 T_{jmi}^L(\mathbf{l}, \mathbf{k} - \mathbf{l}, t) \\ &= \int T_{jmi}(\mathbf{l}, \mathbf{k} - \mathbf{l}, t) d\mathbf{l} \end{aligned} \quad (5.151)$$

where the second equality depends on the fact, implied by Eqs. (5.148) and (5.149), that $\lim_{L \rightarrow \infty} T_{jmi}^L = T_{jmi}$, and that $(2\pi/L)^3$ is the volume surrounding each of the wave number vectors in the sum (since $\mathbf{k} = 2\pi\mathbf{n}/L$). The integral appears in the last term in Eq. (5.151) as the limit of the Riemann sums as $L \rightarrow \infty$. Collecting these results together, it has been established that

$$\begin{aligned} \frac{\partial E(k, t)}{\partial t} + 2\nu k^2 E(k, t) \\ = 2\pi k^2 M_{ijm}(\mathbf{k}) \int (T_{jmi}(\mathbf{l}, \mathbf{k} - \mathbf{l}, t) - T_{jmi}(\mathbf{l}, -\mathbf{k} - \mathbf{l}, t)) d\mathbf{l}. \end{aligned} \quad (5.152)$$

Using the homogeneity properties of $T_{jmi}(\mathbf{l}, -\mathbf{k} - \mathbf{l}, t)$ it can be shown that the second term in the integral on the right-hand side of Eq. (5.152) contributes minus that of the first term so that the equation simplifies to

$$\frac{\partial E(k, t)}{\partial t} + 2\nu k^2 E(k, t) = 4\pi k^2 M_{ijm}(\mathbf{k}) \int T_{jmi}(\mathbf{l}, \mathbf{k} - \mathbf{l}, t) d\mathbf{l}. \quad (5.153)$$

The form of the stretching term in Eq. (5.153) should be contrasted with that derived in Eq. (5.122) incorporating the scalar correlation function $k(r)$.