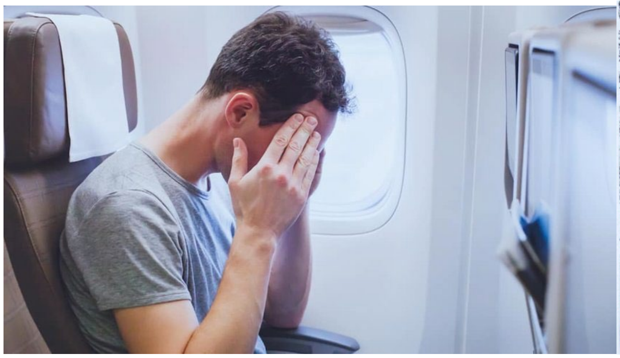
Chapter 1 Introduction

1. Definition of turbulence



It is often claimed that there is no good definition of turbulence (see, *e.g.*, Tsinober [3]), and many researchers are inclined to forego a formal definition in favor of intuitive characterizations. One of the best known of these is due to Richardson [4], in 1922:

> *Big whorls have little whorls,*
> *which feed on their velocity;*
> *And little whorls have lesser whorls,*
> *And so on to viscosity.*

T. von Kármán [5] quotes G. I. Taylor with the following definition of turbulence:

> *"Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or over one another."*

Hinze, in one of the most widely-used texts on turbulence [6], offers yet another definition:

> *"Turbulent fluid motion is an irregular condition of the flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned."*

Chapman and Tobak [1] have described the evolution of our understanding of turbulence in terms of three overlapping eras: *i*) statistical, *ii*) structural and *iii*) deterministic. We shall further explore this viewpoint in the next section, but here we point out that a more precise definition of turbulence is now possible within the context of ideas from the deterministic era. Namely,

> *"Turbulence is any chaotic solution to the 3-D Navier–Stokes equations that is sensitive to initial data and which occurs as a result of successive instabilities of laminar flows as a bifurcation parameter is increased through a succession of values."*

Modern definition superior as (1) specifies equations; (2) requires random behavior described by deterministic equations; (3) requires three dimensionalities; and (4) sensitivity to initial conditions.

2. Historical background

Three eras of turbulence studies: http://web.engr.uky.edu/~acfd/lctr-notes634.pdf

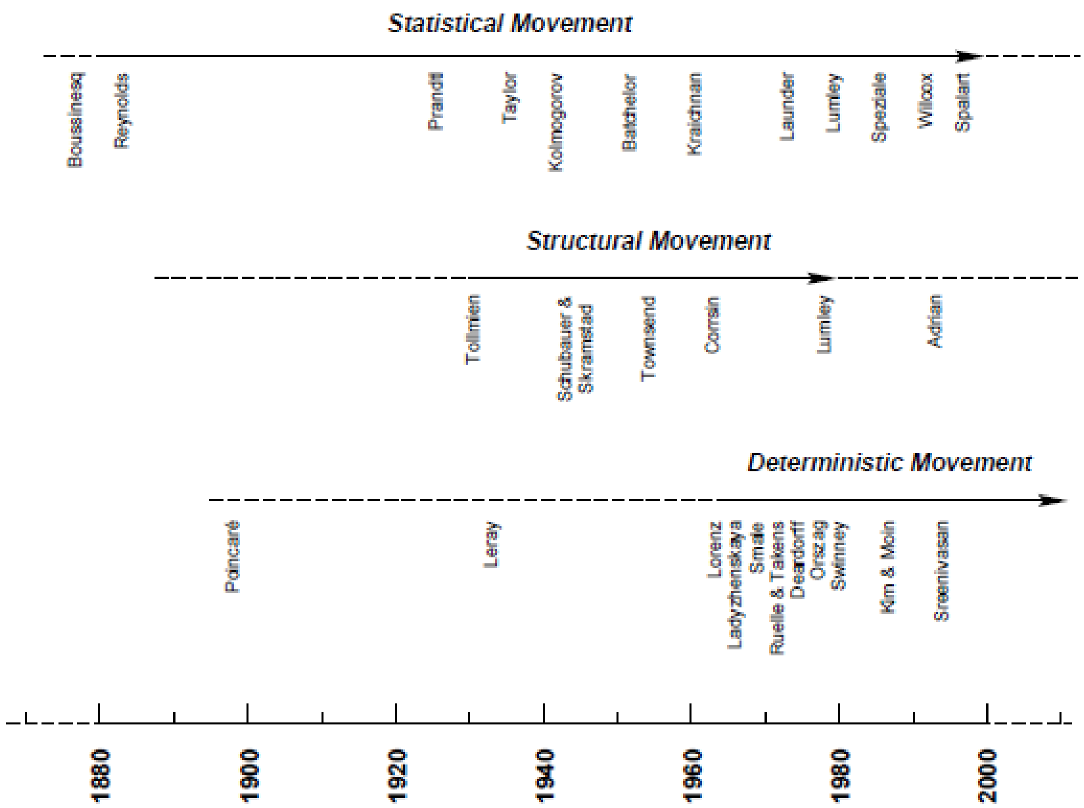


Figure 1.3: Movements in the study of turbulence, as described by Chapman and Tobak [1].

Structural movement: coherent structures and short bifurcation sequences

Statistical movement: uses statistics via systematic approximations to the averaged “unclosed equations” or by intuition and analogy.

Deterministic movement: The idea of the deterministic turbulence has been suggested about a decade ago. In contrast to the usual (random) turbulence, the deterministic turbulent flows have reproducible instantaneous structure, representing one particular-realization from infinite number of possible ones.

3. Syllabus

Course syllabus and semester are divided into Parts 1 and 2 corresponding to 8 weeks up to and 7 weeks after midterm, respectively.

Part 1. Fundamental Concepts: mathematical methods and equations; scaling; isotropic turbulence; and turbulent transport.

Mid-term Exam

Part 2: Canonical turbulent flows (free shear, channel and pipe, and boundary layer) and turbulence modeling

Student Project Presentations

Final Exam

4. Overview Part 1

Chapter 2 Describing turbulence

Chapter 3 Turbulent flow equations

Chapter 4 Turbulence at small scales

Chapter 5 Energy decay in isotropic turbulence

Chapter 6 Turbulent transport and its modeling

Chapter 2 Describing Turbulence

1. NS equations and Reynolds number = $\frac{\text{inertial}}{\text{viscous}}$ forces

2. Averaging: time, ensemble, and phase/conditional

3. One-point statistics: Reynolds decomposition
$\underline{U} = \langle \underline{U} \rangle + \underline{u}$ and Reynolds stresses $= \langle u_i u_j \rangle$

4. Two-point correlations:

Two-point velocity correlation

$R_{ij}(\underline{x}, \underline{y}, t) = \langle u_i(\underline{x}, t)\, u_j(\underline{y}, t) \rangle$

$\underline{x} = \underline{y}$
$R_{ij}(\underline{x}, t) = \langle u_i u_j \rangle$

Two-point triple velocity correlation

$S_{ijk}(\underline{x}, \underline{y}, t) = \langle u_i(\underline{x}, t)\, u_j(\underline{x}, t)\, u_k(\underline{y}, t) \rangle$

Let $\underline{r} = \underline{y} - \underline{x}$ and homogeneous turbulence i.e. $\neq f(\underline{x})$

$R_{ij}(\underline{r}, t) = \langle u_i(\underline{x}, t)\, u_j(\underline{x} + \underline{r}, t) \rangle$

$S_{ijk}(\underline{r}, t) = \langle u_i(\underline{x}, t)\, u_j(\underline{x}, t)\, u_k(\underline{x} + \underline{r}, t) \rangle$

Note: $R_{ij}(0, t) = \langle u_i u_j \rangle$
$R_{ij}(\underline{r} \to \infty, t) = 0$ u_i and u_j uncorrelated

5. Two-point longitudinal & transverse correlation coefficient

$$\overline{u^2} f(r) = R_{11}(r\hat{e}_1)$$

$$\overline{u^2} g(r) = R_{22}(r\hat{e}_1)$$

6 Taylor micro & Macro Scales

$$\lambda_f = \left[-2/f''(0)\right]^{1/2}$$

$$\Lambda_f = \int_0^\infty f(r)\,dr$$

7 Temporal auto correlation coeff & micro & macro scales (based one-point statistics)

$$R_E(\tau) = \langle u(t)\, u(t+\tau)\rangle / \overline{u^2}$$

$$\tau_E = \left[-2/R_E''(0)\right]^{1/2}$$

$$T_E = \int_0^\infty R_E(\tau)\,d\tau$$

$$\lambda_f = \overline{U}\tau_E \quad \Lambda_f = \overline{U}\,T_E$$

Can be transformed spatial via Taylor hypothesis

$$\frac{du_i}{dx} = -\frac{1}{U_c}\frac{du_i}{dt}$$

U_c = convection velocity of frozen turbulence

8 Spatial Spectra

energy (velocity) spectrum tensor

$$\Sigma_{ij}(\underline{k},t) = (2\pi)^{-3}\int_{\forall|_{-\infty}^{\infty}} R_{ij}\, e^{i\underline{k}\cdot\underline{r}}\, d\underline{r} \qquad d\underline{r} = dr_1 dr_2 dr_3$$

$$R_{ij}(\underline{r},t) = \int_{\forall|_{-\infty}^{\infty}} \Sigma_{ij}\, e^{-i\underline{k}\cdot\underline{r}}\, d\underline{k} \qquad d\underline{k} = dk_1 dk_2 dk_3$$

3D Fourier transform pair; implied $f(\underline{x})$
Decompose turbulence correlations into continuous range of scales, or per Fourier modes $e^{i\underline{k}\cdot\underline{r}}$

$tr(R_{ij})/2$
$d r = 0$: spectral decomposition TKE

$$K(t) = \frac{1}{2}\int_{\forall} tr\,\Sigma_{ij}(\underline{k},t)\,d\underline{k} = \frac{1}{2}\int_{\forall} \underbrace{\Sigma_{ii}(\underline{k},t)\,d\underline{k}}_{\text{density energy k space}}$$

$$= \int_0^{\infty} \underbrace{\left[\frac{1}{2}\int_{|\underline{k}|=k} \Sigma_{ii}(\underline{k},t)\,d\Omega\right]}_{\text{energy spectrum}} dk \qquad d\underline{k} = d\Omega\, dk$$

$d\Omega$ = elemental solid angle

$= E(k,t)$ shows how the kinetic energy is distributed across the different scales of the flow ie collect energy shells of fixed distance $k = |\underline{k}|$ from the origin

$$K(t) = \int_0^{\infty} E(k,t)\,dk$$

9. Time Spectra

Fourier Transform pair

$$\hat{R}_E(\omega') = \int_{-\infty}^{\infty} e^{-i\tau\omega'} R_E(\tau)\, d\tau \qquad R_E(\tau) = \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2}}$$

$$R_E(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\omega'} \hat{R}_E(\omega')\, d\omega' \qquad \omega' = 2\pi\omega \ \frac{\text{rad}}{s}$$

ω hz

$$R_E(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}_E(\omega')\, d\omega' = 1 = \int_{-\infty}^{\infty} \underbrace{\hat{R}_E(2\pi\omega)}_{=\hat{E}_{11}(\omega)/2\overline{u^2}}\, d\omega$$

$$\int_{-\infty}^{\infty} \frac{\hat{E}_{11}(\omega)}{2\overline{u^2}}\, d\omega = 1$$

define 1D energy spectrum

$$\overline{u^2} = \frac{1}{2}\int_{-\infty}^{\infty} \hat{E}_{11}(\omega)\, d\omega$$

for stationary flow: $u(t)u(t+\tau) = u(t-\tau)u(t)$

$$R_E(\tau) = R_E(-\tau)$$

$$\Rightarrow \hat{R}_E(\omega') = 2\int_0^{\infty} \cos\tau\omega'\, R_E(\tau)\, d\tau$$

$$\hat{R}_E(-\omega') = \hat{R}_E(\omega')$$

$$R_E(\tau) = \frac{1}{\pi}\int_0^{\infty} \cos\tau\omega'\, \hat{R}_E(\omega')\, d\omega'$$

$$\overline{u^2} = \int_0^{\infty} \hat{E}_{11}(\omega)\, d\omega$$

Note: $R_E(\tau)$ thus $\hat{E}_{11}(\omega)$ can be obtained single point measurement vs $E(\kappa, t)$ requires volume or line measurements and if Taylor frozen turbulence hypothesis used can be transformed from time to space as approximation for Spatial Spectra.

Chapter 3 Turbulent flow equations

Instantaneous equations

- Continuity and Navier -Stokes equations
- Mechanical energy equation
- Energy equation
- Vorticity equation
- Enstrophy equation
- Pressure equation

Reynolds averaged equations

- Continuity and RANS
- KE mean flow equation
- TKE equation
- Dissipation ε equation
- Reynolds stress equation
- Mean vorticity equation
- Fluctuating vorticity equation
- Vorticity transport equation
- Enstrophy equation

Chapter 4 Turbulence at Small Scales

1. Energy cascade & Kolmogorov hypotheses

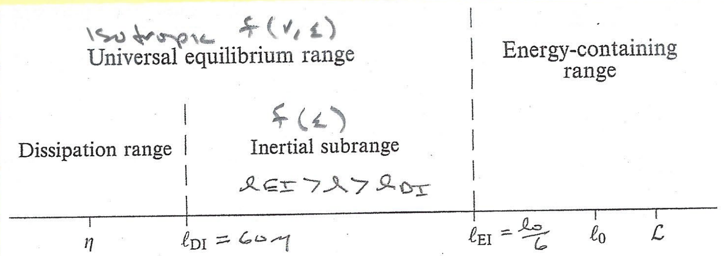


Fig. 6.1. Eddy sizes ℓ (on a logarithmic scale) at very high Reynolds number, showing the various lengthscales and ranges.

$$\eta = (\nu^3/\varepsilon)^{1/4} \qquad u_\eta = (\varepsilon\nu)^{1/4} \qquad \tau_\eta = (\nu/\varepsilon)^{1/2}$$

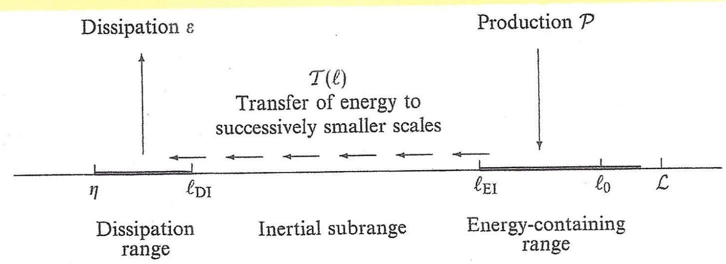


Fig. 6.2. A schematic diagram of the energy cascade at very high Reynolds number.

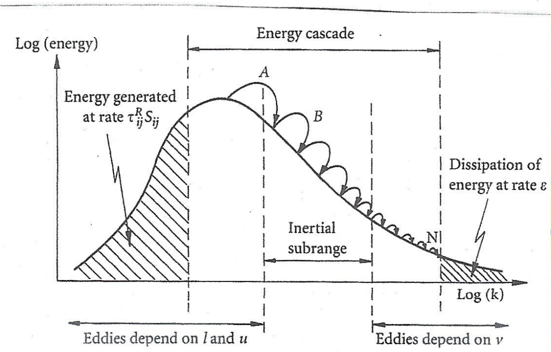


Figure 5.3 Schematic representation of the energy cascade.

2 Spectral representation ε

$$\langle u_{i,k}\, u_{j,l} \rangle = \frac{\partial^2 R_{ij}}{\partial r_k \partial r_l}(0)$$

$$\langle u_{i,k}\, u_{i,k} \rangle = \frac{\partial^2 R_{ii}}{\partial r_l^2}(0) = \varepsilon / \nu$$

$$\varepsilon = 2\nu \int_0^\infty k^2 E(k)\, dk \qquad k = \int_0^\infty E(k)\, dk$$

motions leading to K and ε come from different wave number ranges k_ε & k_d

energy containing eddies size $l_e = 1/k_\varepsilon$ vs dissipation eddies size $l_d = 1/k_d$

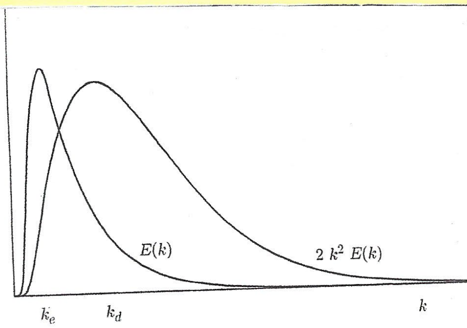


Figure 4.1 Spectral ranges of $E(k)$ and $2k^2E(k)$, with k_e and k_d marking their respective peaks.

Physics of transfer process i.e. energy cascade fundamental to theory of turbulence. Separation scales increases with Re & for sufficiently high Re transfer occurs without dissipation at intermediate scales

3 Consequences of isotropy

Isotropy & definitions $f(r)$ & $g(r)$

$$R_{ij}(\underline{r}) = \overline{u^2}\left[(f-g)\frac{r_i r_j}{r^2} + g\,\delta_{ij}\right] \qquad f=f(r),\ g=g(r)$$

Similar analysis for S

$$S_{ijl}(\underline{r}) = u_{rms}^3\left[(k-h-2q)\frac{r_i r_j r_l}{r^3} + \delta_{ij}\, h\,\frac{r_l}{r} + q\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

$$S_{111}(r\hat{e}_1) = u^3_{rms}\, k(r)$$

$$S_{221}(r\hat{e}_1) = u^3_{rms}\, h(r)$$

$$S_{212}(r\hat{e}_1) = u^3_{rms}\, q(r)$$

Further simplifications using $\nabla\cdot\underline{u}=0$ & various identities

$$R_{ij}(\underline{r}) = \overline{u^2}\left[\left(f + \frac{r}{2}f'\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}f'\right]$$

$$S_{ijl}(\underline{r}) = u_{rms}^3\left[(k - r k')\frac{r_i r_j r_l}{2r^3} - \frac{k}{2}\delta_{ij}\frac{r_l}{r} + \frac{1}{4r}(k r^2)'\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

Shows R_{ij} & S_{ijl} are only functions f & k, respectively, & both are scalars. Both will be used extensively for study of homogeneous isotropic turbulence.

4 The smallest scales

$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ $\quad t_\lambda = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$ $\quad v_\lambda = \eta / t_\lambda = (\nu\varepsilon)^{1/4}$

$= 1/k_d$

$\varepsilon = \frac{\nu}{2}\langle u_{i,j} + u_{j,i}\rangle^2 = \nu\langle u_{i,j}^2 + u_{i,j}u_{j,i}\rangle$

$= \tilde{\varepsilon} + \nu \frac{\partial^2 \langle u_i u_j\rangle}{\partial x_i x_j}$

$= \tilde{\varepsilon}$ for isotropic turbulence

$\langle u_{i,j} u_{k,l}\rangle = 4^{th}$ order tensor

isotropic $= \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$

$\varepsilon = \nu\langle u_{i,j}^2\rangle = \frac{15}{2}\nu\beta = 15\nu\langle u_x^2\rangle$

$\varepsilon/\nu = -\frac{\partial^2 R_{ii}}{\partial r_k^2}(0) = -15\overline{u^2} f''(0)$ $\quad \overline{u^2} f''(0) = -\langle u_x^2\rangle$

$\lambda_f^2 = -2/f''(0)$

$\varepsilon = -15\nu\overline{u^2} f''(0) = 30\nu\overline{u^2}/\lambda_f^2$

$= 15\nu\overline{u^2}/\lambda_g^2$ $\quad \lambda_f = \sqrt{2}\lambda_g$

$u_{rms} = [\overline{u^2}]^{1/2}$ $\quad l_e \sim L$ $\quad t_e =$ eddy turn over time

$= l_e/u_{rms}$

$\varepsilon = \frac{u_{rms}^2}{t_e} = \frac{u_{rms}^3}{l_e} \Rightarrow \frac{\nu u_{rms}^2}{\lambda} \sim \frac{u_{rms}^3}{l_e}$ or $\frac{l_e}{\lambda} \sim R_\lambda = \frac{u_{rms}\lambda}{\nu}$

$R_\lambda \sim 100 \Rightarrow 10^2$ difference l_e and λ

$Re^{1/2} \sim R_\lambda$ $\quad Re = u_{rms} l_e/\nu$

(5) inertial subrange

using dimensional analysis $E(\kappa)$ units m^3/s^2

universal equilibrium range velocity statistics $f(\varepsilon, \nu)$: $E(\kappa) = (\varepsilon \nu^5)^{1/4}\, \varphi(\kappa\eta)$

ε, ν nondimensionalize κ & $E(\kappa)$ $\qquad$ $\varphi(\kappa\eta)$ = Kolmogorov spectrum function

or $E(\kappa) = \varepsilon^{2/3} \kappa^{-5/3}\, \Psi(\kappa\eta)$

ε, κ nondimensionalize $E(\kappa)$ $\qquad$ Ψ compensated Kolmogorov spectrum function

$$\Psi(\kappa\eta) = (\kappa\eta)^{5/3}\, \varphi(\kappa\eta)$$

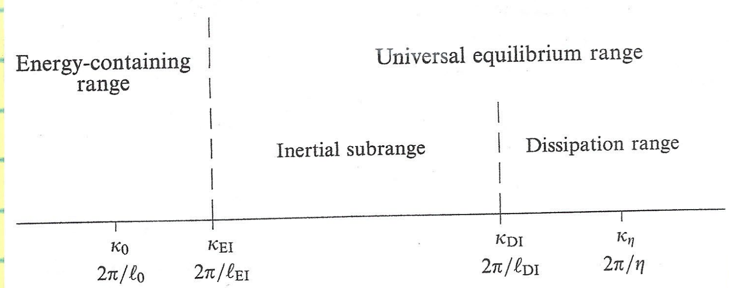


Fig. 6.12. Wavenumbers (on a logarithmic scale) at very high Reynolds number showing the various ranges.

inertial subrange $f(\varepsilon)$ only & ν only via η : $\Psi(\kappa\eta) \to C$ = constant

$$E(\kappa) = C\, \varepsilon^{2/3} \kappa^{-5/3}$$

$$E_{11}(\kappa_1) = C_1\, \varepsilon^{2/3} \kappa_1^{-5/3} \qquad \text{1D spectra}$$

Kolmogorov $-5/3$ spectrum!

6. Relationship 1D and 3D Spectra
7. 1D Spatial and Temporal Spectra

Method 1: Energy Spectrum from space autocorrelation f(r): even

1. Calculate symmetric space autocorrelation function
$$f(r) = \frac{\langle u(x)u(x+r)\rangle}{\langle u^2\rangle}$$

2. Obtain 1D energy spectrum from Fourier transform of $f(r)$
$$E_{11}(k_1) = \frac{2}{\pi}\langle u^2\rangle \int_0^\infty f(r_1)\cos(k_1 r_1)\,dr_1$$

3. Calculate the Taylor microscale and integral length scale
$$\lambda_f = \left[-2/f''(0)\right]^{1/2},\ \Lambda_f = \frac{1}{2}\int_{-\infty}^{\infty} f(r)dr = \int_0^\infty f(r)dr$$

4. Calculate dissipation
$$\varepsilon = 30\nu\langle u^2\rangle/\lambda_f^{\,2}$$

5. Calculate Kolmogorov scale
$$\eta = \left(\nu^3/\varepsilon\right)^{1/4}$$

6. Plot $E_{11}(k_1)/\left(\varepsilon\nu^5\right)^{1/4}$ vs $k_1\eta$

Method 2: Energy Spectrum from space autocorrelation f(r): odd

1. Calculate the antisymmetric space autocorrelation function
$$f(\pm r) = \frac{\langle u(x)u(x\pm r)\rangle}{\langle u^2\rangle}$$

2. Obtain 1D energy spectrum in space from Fourier transform of $f(r)$

$$E_{11}(k_1)=\frac{\langle u^2\rangle}{\pi}\int_{-\infty}^{\infty} f(r_1)\cos(k_1 r_1)\,dr_1$$

3. Calculate the Taylor microscale and integral length scale

$$\lambda_f=\left[-f'(0)+\left[\{f'(0)\}^2-2f''(0)\right]^{1/2}\right]\Big/ f''(0),\ \Lambda_f=\frac{1}{2}\int_{-\infty}^{\infty} f(r)dr$$

4. Calculate dissipation

$$\varepsilon=30\nu\langle u^2\rangle/\lambda_f^{\,2}$$

7. . Calculate Kolmogorov scale

$$\eta=\left(\nu^3/\varepsilon\right)^{1/4}$$

5. Plot $E_{11}(k_1)\big/\left(\varepsilon\nu^5\right)^{1/4}$ vs $k_1\eta$

Method 3: Method 2 + direct calculation of dissipation

In Step 5 of Method 2, Use the equation for direct calculation

$$\varepsilon=\nu\left\langle\left(\frac{\partial u_i}{\partial x_j}\right)^2+\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\rangle$$

Method 4: Energy spectrum from temporal autocorrelation f(τ)

1. Calculate temporal autocorrelation

$$R_E(\tau)=\frac{\langle u(t)u(t+\tau)\rangle}{\langle u^2\rangle}$$

2. Obtain Fourier transform of $R_E(\tau)$

$$\hat{R}_E(2\pi\omega)=2\int_0^{\infty} R_E(\tau)\cos(2\pi\omega\tau)\,d\tau$$ (Note: ω: Frequency [Hz])

3. Calculate the time scale and integral time scale

$$\tau_E = \left[-2/f''(0)\right]^{1/2}, \mathrm{T} = \int_0^\infty f(\tau)d\tau$$

4. Calculate the Taylor microscale, dissipation and Kolmogorov scale

$$\lambda_f = \bar{U}\tau_E, \ \varepsilon = 30\nu\left\langle u^2\right\rangle / \lambda_f^{\,2}, \eta = \left(\nu^3/\varepsilon\right)^{1/4}$$

5. Calculate the 1D energy spectrum in time from the Fourier transform of $R_E(\tau)$

$$\hat{E}_{11}(\omega) = 2\left\langle u^2\right\rangle \hat{R}_E(2\pi\omega)$$

6. Calculate the 1D energy spectrum in space from the 1D energy spectrum in time

$$E_{11}(k_1) = \frac{\bar{U}}{2\pi}\hat{E}_{11}(\omega)$$

7. Plot $E_{11}(k_1) / \left(\varepsilon\nu^5\right)^{1/4}$ vs $k_1\eta$

$$E_{11}(k_1) = \frac{2}{\pi}\overline{u^2}\int_0^\infty f(r_1)\cos(k_1 r_1)\,dr_1$$

$$= \iint_{-\infty}^{\infty} \frac{E(k)}{2\pi k^2}\left(1 - \frac{k_1^2}{k^2}\right) dk_2\,dk_3$$

$$= \int_{k_1}^{\infty} \frac{E(k)}{k}\left(1 - \frac{k_1^2}{k^2}\right) dk$$

$$E(k) = \frac{1}{2}k^3\frac{d}{dk}\left(\frac{1}{k}\frac{dE_{11}(k)}{dk}\right)$$

Dissipation Spectra

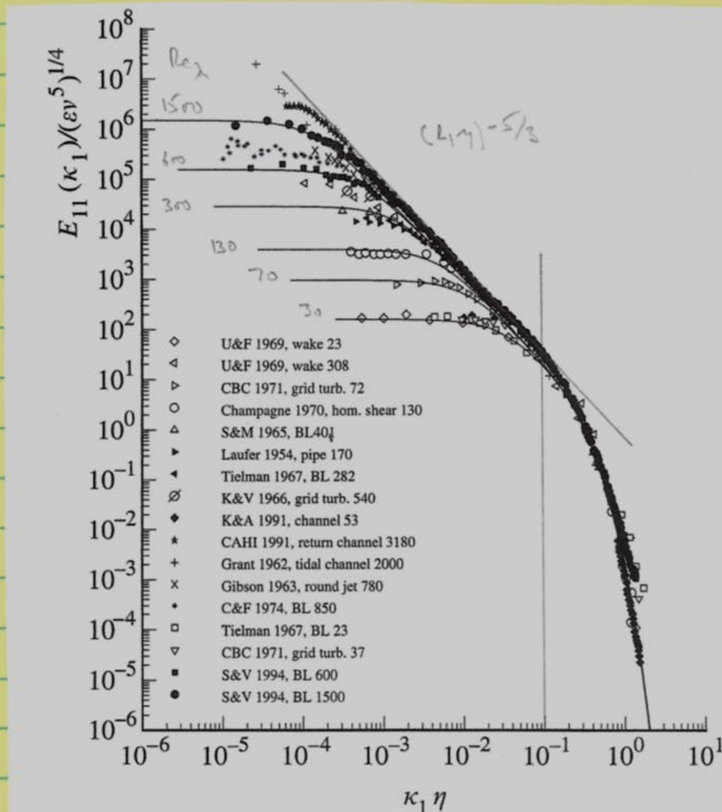


Fig. 6.14. Measurements of one-dimensional longitudinal velocity spectra (symbols), and model spectra (Eq. (6.246)) for $R_\lambda = 30, 70, 130, 300, 600$, and 1,500 (lines). The experimental data are taken from Saddoughi and Veeravalli (1994) where references to the various experiments are given. For each experiment, the final number in the key is the value of R_λ.

$E_{11}(k_1)/(\varepsilon\nu^5)^{1/4} = \Phi_{11}(k_1\eta)$

* universal $f(k_1\eta)$ for high Re & $k_1 > k_{EI}$: universal equilibrium range
* Data lie on single curve for $k_1\eta > .1$: dissip range
* Power law for $k_1\eta < .1$ & extent region increases Re_λ : inertial subrange
* model spectrum accurate

8 Structure functions

$$S_n(\underline{x},\underline{r},t) = \langle|\underline{u}(\underline{x}+\underline{r},t) - u(\underline{x},t)|^n\rangle$$

n^{th} order difference in $\underline{u}$ between $\underline{x}+\underline{r}$ & $\underline{x}$

small r dissipation range & larger r inertial subrange

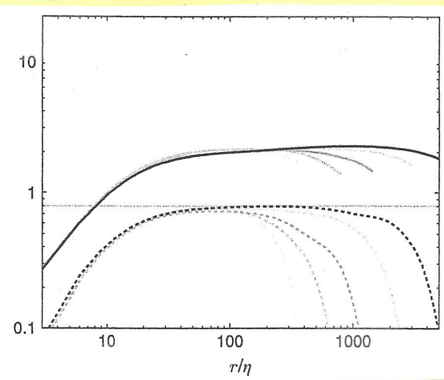


Figure 4.10 Compensated longitudinal structure functions computed in isotropic turbulence [24]. $S_2^L/(\epsilon_r r)^{2/3}$, top curves; $-S_3^L/(\epsilon_r r)$, bottom curves. Constant, dotted line is 0.8. The sequence of curves in each group covers an increasing r/η domain correspond to $R_\lambda = 167, 257, 471, 732, 1131$.

9 Pope

Chapter 5 Energy decay in isotropic turbulence

Chapter 5 Energy Decay in Isotropic Turbulence

Physical domain = large & away boundaries
or box with periodic BC

$f(r)$ & $h(r) \to 0$ well within domain

Provides of h & λ_f, $\to \Phi$ requiring & Φ

IC: homogeneous/isotropic turbulence
at $t \to$ no flow ; K, ε, etc. @ 0
given along with
$f(r)$ & $h(r)$
$\Rightarrow \lambda_f$

All variables $f(t)$

(3.38) $K_t = -\varepsilon$

(3.42) $\varepsilon_t = P_\varepsilon^d - Y_\varepsilon = -2\nu\langle u_{i,k} u_{i,j} u_{k,j}\rangle - 2\nu^2 \langle \frac{\partial^2 u_i}{\partial x_j \partial x_k}^2 \rangle$

or $\varepsilon = \nu S$

$S_t = \nu P_S^\nu - \nu Y_S = 2\nu\langle \omega_i \omega_k u_{i,k}\rangle - 2\nu^2 \langle \omega_{i,k} \omega_{i,k}\rangle$

Solution provides decay K & rate ε
and evolution ε as per RHS terms:
P terms = vortex stretching = Production $\varepsilon > 0$
Y terms = dissipation of ε

$$\langle (u_{i,i})^2_{,\ell} \rangle = \frac{\partial^4 R_{ii}}{\partial r_1^2 \partial r_\ell^2}(0) = 35\,\overline{u^2}\, f^{iv}(0)$$

$$\langle u_{i,\ell}\, u_{\ell,i}\, u_{i,i} \rangle = \frac{\partial^3 S_{i\ell i}}{\partial r_\ell \partial r_1^2}(0) = \frac{35}{2}\, u^3_{rms}\, k'''(0)$$

$$\varepsilon_t = -35\nu \underbrace{u^3_{rms}\, k'''(0)}_{\langle u_{1,1}^3\rangle} - 70\nu^2 \underbrace{\overline{u^2}\, f^{iv}(0)}_{\langle (u_{1,1})^2_{,1}\rangle}$$

$u_{rms} = \sqrt{\overline{u^2}}$

$u^2_{rms} = \overline{u^2}$

$\left\langle \frac{\partial u^3}{\partial x} \right\rangle$ $\qquad$ $\langle u_{xx}^2 \rangle$

$u^3_{rms}\, k''(0) = \langle u_x^3 \rangle$ $\qquad$ S = skewness u_x = $-\dfrac{\langle u_x^3\rangle}{\langle u_x^2\rangle^{3/2}}$

$k''(0) = \langle u_x^3 \rangle / u^3_{rms}$

$= -S/\lambda_g^3$ $\qquad$ $-S\langle u_x^2\rangle^{3/2} = \langle u_x^3\rangle$

$\langle u_x^2 \rangle = u^2_{rms}/\lambda_g^2$

$= -S\left(\dfrac{\varepsilon}{15\nu\overline{u^2}}\right)^{3/2}$ $\qquad$ $\langle u_x^2\rangle^{3/2} = u^3_{rms}/\lambda_g^3$

$\langle u_{xx}^2 \rangle = u^2_{rms}\, f^{iv}(0)$ $\qquad$ $\varepsilon = 15\nu\overline{u^2}/\lambda_g^2$

$\lambda_g^2 = 15\nu\overline{u^2}/\varepsilon$

$G = \overline{u^2}\langle u_{xx}^2\rangle / \langle u_x^2\rangle^2$ $\qquad$ $\lambda_g^3 = \left(15\nu\overline{u^2}/\varepsilon\right)^{3/2}$

= palenstrophy

$f^{iv}(0) = \langle u_{xx}^2 \rangle / u^2_{rms}$ $\qquad$ $G\langle u_x^2\rangle^2/\overline{u^2} = \langle u_{xx}^2\rangle$

$\dfrac{G\, u^2_{rms}}{\lambda_g^4} = \langle u_{xx}^2 \rangle$

$= G/\lambda_g^4$

$= G\left(\dfrac{\varepsilon}{15\nu\overline{u^2}}\right)^2$

$u_{rms} = (\overline{u^2})^{1/2}$

$u^3_{rms} = \overline{u^2}^{3/2}$

$$\varepsilon_t = -35 \nu u^3_{rms}\left[-S\left(\frac{\varepsilon}{15\nu\overline{u^2}}\right)^{3/2}\right] - 70\nu^2 \overline{u^2}\left[G\left(\frac{\varepsilon}{15\nu\overline{u^2}}\right)^2\right]$$

$K = \frac{3}{2}\overline{u^2}$

$\frac{2}{3}K = \overline{u^2}$

$$-35\nu\,\overline{u^2}^{3/2}\left(-S\left(\frac{\varepsilon}{15\nu\overline{u^2}}\right)^{3/2}\right) - \frac{70\nu^2}{\overline{u^2}}\, G\, \frac{\varepsilon^2}{(15\nu)^2}$$

$$+35\nu S \frac{\varepsilon^{3/2}}{(15\nu)^{3/2}}$$

$$-\frac{70}{\frac{2}{3}K}\, G\, \frac{\varepsilon^2}{15^2} = -\frac{7}{15} G \frac{\varepsilon^2}{K}$$

$$\frac{5\times 7}{15 \cdot 15^{1/2}} = \frac{7}{3\sqrt{15}} \qquad \frac{35 S\, \varepsilon^{3/2}}{15^{3/2}\nu^{1/2}}$$

$$\frac{7}{3\sqrt{15}}\,\frac{K}{\nu^{1/2}\varepsilon^{1/2}}\,\frac{\varepsilon^2}{K} = \frac{7}{3\sqrt{15}}\cdot\frac{1}{\nu^{1/2}}\,\varepsilon^{3/2}$$

$$\varepsilon_t = S_K^* R_T^{1/2}\, \varepsilon^2/K - G^* \varepsilon^2/K \quad \wedge \quad K_t = -\varepsilon$$

$S_K^* = \frac{7}{3\sqrt{15}} S_K$

$G^* = \frac{7}{15} G$

$R_T = \frac{K^2}{\nu\varepsilon}$

2 equations

4 unknowns = K, ε

S_K^*, G^*

IC: $t=0$ $K_0, \varepsilon_0, S_{K0}^*, G_0^*$

or $f(\nu)$

$\wedge R(\nu)$

at $t=0$

R_T = turbulent Re = $\frac{VL}{\nu}$ $V = \sqrt{K}$ $L = K^{3/2}/\varepsilon$

$T_t = K/\varepsilon$ = eddy turnover time = time scale for significant fraction TKE dissipate

$T_t^{-1} = \varepsilon/K = -K^{-1} K_t$

$R_T = \frac{K/\varepsilon}{\nu/K} = T_t/T_\mu$ $T_\mu = \nu/K$ = time scale for viscous dissipation

$R_T \uparrow$ energetic turbulence $T_t >> T_\mu$

$R_T \downarrow 0$ final stage decay

R_T in vortex stretching term P^4_ε is significant when turbulence is energetic

Note: $R_T = 3/20\, R_\lambda^2$

$R_\lambda < 1$ final period decay

$R_\lambda > 100$ not weak

$= 1000$ strong

Interest: large $R_T \rightarrow R_T < 1$

$\frac{dK}{dt} = -\varepsilon$ and $\frac{d\varepsilon}{dt} = S_K^* R_T^{1/2} \varepsilon^2/K - G^* \varepsilon^2/K$

Can be combined: $\frac{dR_T^*}{d\tau} = R_T^* (G^* - 2 - S_K^* \sqrt{R_T^*})$

1 eq 3 unknowns
+ IC

$\tau(t) = \ln[K(0)/K(t)]$

$R_T^* = R_T(t(\tau))$

Modes of anisotropic decay

Self similarity : $f(r,t) = \bar{f}(r/L(t))$ $\Rightarrow$ Sk & G
$h(r,t) = \bar{h}(r/L(t))$ constant

length scale $L(t)$
Changes with time, but overall structure
multiple point correlations retains form
of independent time, which leads
to conflict since large separation
r of λ.

$K_t = -\varepsilon \quad \varepsilon_t = S\hat{k}_0^* R_T^{1/2} \varepsilon^2/K - G_0^* \varepsilon^2/K$ 2 eq
2 unknowns

or $R^*_{T\,t} = R_T^* \left(G_0^* - 2 - S\hat{k}_0^* \sqrt{R_T^*}\right)$

Fixed point analysis : $\frac{dR_T^*}{dt} = 0$ attraction solutions
$t \to \infty$: equilibrium
states decay

$R^*_{T\infty}\left(G_0^* - 2 - S\hat{k}_0^* \sqrt{R^*_{T\infty}}\right) = 0$ isotropic decay

$R^*_{T\infty} = 0$ or $R^*_{T\infty} = \left(\frac{G_0^* - 2}{S\hat{k}_0^*}\right)^2$

$G_0^* \leq 2$ $\qquad$ $G_0^* > 2$

R_T^* decay to 0 $\qquad$ R_T^* go $R^*_{T\infty}$ above or below depending R_{T0}

Final period isotropic decay : $R^*_{T\infty} = 0$

$$\varepsilon_t = \left[\frac{S_K^* R_T^{1/2}}{G_0^*} - 1 \right] G_0^* \, \varepsilon^2/K$$

$\ll 1$ if neglect vortex stretching

Low Re EFD/DNS : $S_K \approx .5$ $\quad G \approx 3 \Rightarrow S_K^* \approx .3$

$G_0^* \approx 1.4 < 2$

$\varepsilon_t = -G_0^* \, \varepsilon^2/K \qquad K_t = -\varepsilon \qquad \dfrac{S_K^* R_T^{1/2}}{G_0^*} \ll 1 \Rightarrow R_T < .1$

$K/K_0 = \left(1 + \dfrac{t}{\alpha T_{\varepsilon 0}}\right)^{-\alpha} \qquad \varepsilon/\varepsilon_0 = \left(1 + \dfrac{t}{\alpha T_{\varepsilon 0}}\right)^{-1-\alpha}$

$\alpha = 1/(G_0^* - 1) \qquad K_0, \varepsilon_0 \quad T_{\varepsilon 0} = K_0/\varepsilon_0$

for $t/(\alpha T_{\varepsilon 0}) \gg 1$: $K \sim t^{-\alpha}$ $\quad$ power laws

$\varepsilon \sim t^{-1-\alpha} \qquad \alpha = \alpha(G_0^*)$

EFD & theory : $f(r,t) = e^{-r^2/2\lambda_g^2}$ $\Rightarrow G = 3$ & $G_0^* = 7/5$

$\alpha = 5/2 \qquad K \sim t^{-5/2}$

Recall : $\lambda_g^2 = \dfrac{15\nu \overline{u^2}}{\varepsilon}$ $\qquad$ agrees EFD

$= \dfrac{10\nu K}{\varepsilon} \qquad K = 3/2\,\overline{u^2}$

ie $\lambda^2 \sim t \qquad K/\varepsilon = \dfrac{t^{-\alpha}}{t^{-1-\alpha}} \sim t$

$\lambda \sim \sqrt{t}$

Small scale motions decay faster large scale motions such that scale of surviving turbulence increases : $f(r) \to 1$ $\quad$ laminar

$\lambda \to \infty$ $\quad$ state

High Re equilibrium : $G_0^* > 2$

$$\frac{d\epsilon}{dt} = (G_0^* - 2)\epsilon^2/K - G_0^*\epsilon^2/K = -2\,\epsilon^2/K$$

$$R_{T\infty}^* = \left(\frac{G_0^* - 2}{S_{K_0}^*}\right)^2 \qquad S_{K_0}^2 = (G_0^* - 2)^2/R_{T\infty}'$$

$$S_{K_0} = (G_0^* - 2)/R_{T\infty}^{*1/2}$$

$$\epsilon_t = \frac{(G_0^* - 2)}{\sqrt{R_{T\infty}^*}} R_T^{1/2} \frac{\epsilon^2}{K} - G_0^*\,\epsilon^2/K$$

for $R_T^{1/2} \to \sqrt{R_{T\infty}^*}$ $\qquad R_{T\infty} >> 1 \Rightarrow G_0^* \sim 25$

typical S_K^*

$K \sim t^{-1}$ $\quad \epsilon \sim t^{-2}$ $\qquad \Rightarrow$ balance vortex

not $f(IC)$ $\qquad$ stretching of Y_ϵ

$R_T \to R_{T\infty}$ in few eddy turn over times

Implications turbulence models

$\Uparrow$

K, ϵ

Equation for two point correlations

but no information flow structure yet S, G, which were assumed constant. Equation for $R_{ij}(\underline{x}, \underline{y}, t)$: $\langle u_i(\underline{x},t)\, NS_j(\underline{y}) + u_j(\underline{x},t)\, NS_i(\underline{y})\rangle$

incompressible flow + homogeneous / isotropic

$$R_{ij,t}(\underline{r},t) = S_{jki,r_k}(-r,t) + S_{ikj,r_k}(r,t)$$

$$- \tfrac{1}{2} K_{i,r_j}(r,t) - \tfrac{1}{2} K_{j,r_i}(-r,t)$$

$$+ 2\nu R_{ij,r_k r_k}(r,t)$$

$K_i(\underline{x}, \underline{y}, t) = \langle u_i(\underline{x},t)\, p(\underline{y},t)\rangle$

$$\underbrace{R_{ii,t}(\underline{r},t) = S_{i,ii,r_2}(-r,t) + S_{i,ii,r_2}(\underline{r},t)}_{\text{vortex stretching}}$$

$$\underbrace{+\,2\nu R_{ii,r_2r_2}(\underline{r},t)}_{\text{viscous diffusion}}$$

Pope 201-206 Karman Howarth equation using R_{ij} & S_{ijk} equations

$$\frac{\partial}{\partial t}(\overline{u^2}f) = \overline{u^2}^{3/2}\left(k_r + \frac{4}{r}k\right) + 2\nu\overline{u^2}\left(f_{rr} + \frac{4}{r}f_r\right) \qquad f(r,t),\ k(r,t)$$

Substitute TS for f & k + $\varepsilon = \frac{15\nu\overline{u^2}}{\lambda_g^2}$ $\quad u_{rms} = \sqrt{\overline{u^2}}$

$\Rightarrow$ $K_t = -\varepsilon$ & $\varepsilon_t = S_k^* R_T^{1/2}\varepsilon^2/K - G^*\varepsilon^2/K$ shows some

Self similar assumption information both approaches

(1) $R_T = 0$ $\Rightarrow$ $\tilde{f}(\eta) = e^{-\eta^2/2}$ $\quad \eta = r/\lambda_g$ $\quad e^{-x^2} = 1 - x^2 + \frac{x^4}{2!}$

$$\tilde{k}(\eta) = \frac{7}{6}S_{k_0}\eta^{-4}\left[(\eta^5 + 5\eta^3 + 15\eta)e^{-\eta^2/2} - 15\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right]$$

slower decay $\tilde{f}(\eta)$ $\quad \tilde{k}(\eta) \sim \eta^{-4}$ as $\eta \to \infty$

(2) R_T large more difficult solution KH equation ie one equation & two unknowns!

Energy Spectrum equation

energy spectrum tensor

FFT (R_{ij} equation) $\quad \Sigma_{ij}(\underline{k},t) = (2\pi)^{-3} \int_{\forall} e^{i\underline{k}\cdot\underline{r}} R_{ij}(\underline{r},t)\, d\underline{r}$

$$\frac{\partial \Sigma_{ij}}{\partial t}(\underline{k},t) = T_{ij}(\underline{k},t) + P_{ij}(\underline{k},t) - 2k^2\nu\,\Sigma_{ij}(\underline{k},t)$$

$R_{ij}(\underline{r},t) = \int_{\forall} e^{-i\underline{k}\cdot\underline{r}} \Sigma_{ij}(\underline{k},t)\, d\underline{k}$

two point velocity correlation tensor

T_{ij} = gain/loss energy at $\underline{k}$ due transfer other scales (vortex stretching or reorientation)

P_{ij} = transport pressure velocity term
= 0 isotropic turbulence
pressure effects anisotropy → isotropy

last term = $2\nu k^2 \Sigma_{ij}$ = rate viscous dissipation

Contract indices $\quad E(k,t) = \int_{|\underline{k}|=k} \Sigma_{ii}(\underline{k},t)\, d\Omega$

$$\frac{\partial E}{\partial t} = T(k,t) - 2\nu k^2 E(k,t) \qquad T(k,t) = \frac{1}{2}\int_{|\underline{k}|=k} T_{ii}(\underline{k},t)\, d\Omega$$

① $E(k) = \frac{\overline{u^2}}{\pi}\int_0^\infty (3f + rf')\, kr \sin kr\, dr \qquad$ (4.73) ⇒ E if f known

② ⇒ T $\qquad$ Say $f = e^{-r^2/2\lambda_g}$

$$E = \frac{u^2_{rms}\,\lambda_g}{\sqrt{2\pi}} (k\lambda_g)^4 e^{-1/2(k\lambda_g)^2}$$

$$T(k,t) = E(k,t)\,\frac{u_{rms}}{\lambda_g}\left((k\lambda_g)^2 - 5\right)$$

$k < \sqrt{5}/\lambda_g$ lose energy to $k > \sqrt{5}/\lambda_g$

Energy spectrum equation via velocity field

$$u_i(\underline{x},t) = \sum_{\underline{k}} \hat{u}_i(\underline{k},t)\, e^{i\underline{k}\cdot\underline{x}} \qquad \underline{k} = 2\pi\underline{n}/L$$

$$\underline{n} = (n_1, n_2, n_3)$$

Pope (6.146) (*)

$$\frac{\partial \hat{u}_j}{\partial t}(\underline{k},t) + \nu k^2 \hat{u}_i(\underline{k},t) = -i P_{ij}(\underline{k})\, k_m \sum_{\underline{l}} \hat{u}_j(\underline{l},t)\,\hat{u}_m(\underline{k}-\underline{l},t)$$

$$P_{ij}(\underline{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}$$ = projection operator maps any vector onto components ⊥ $\underline{k}$

$$E_{\underline{k}} = |\hat{\underline{u}}(\underline{k},t)|^2/2$$

$$\langle (*)\, \underbrace{\hat{u}_i^*}_{\text{complex conjugate } \hat{u}_i}(\underline{k},t) + \text{complex conjugate } (*)\, \hat{u}_i(\underline{k},t) \rangle$$

$$\frac{\partial E_{\underline{k}}}{\partial t} + 2\nu k^2 E_{\underline{k}} = \frac{1}{2} M_{ijm}(\underline{k}) \sum_{\underline{l}} \langle \hat{u}_i(-\underline{k})\hat{u}_j(\underline{l})\hat{u}_m(\underline{k}-\underline{l})\rangle - \langle \hat{u}_i(\underline{k})\hat{u}_j(\underline{l})\hat{u}_m(-\underline{k}-\underline{l})\rangle$$

Continuous form:

$$\frac{\partial E}{\partial t} + 2\nu k^2 E = 4\pi k^2 M_{ijm}(\underline{k}) \int T_{jmi}(\underline{l}, \underline{k}-\underline{l}, t)\, d\underline{l}$$

Shows energy transfer between k, especially note triadic nature transfer!

Chapter 6 Turbulent transport and its modeling

Chapter 6 Turbulent Transport & its Modeling

Molecular Momentum Transport

1. mixing time
2. momentum preserved
3. v linear over mixing length

$\sigma_{12} = \mu \bar{U}_y = \underbrace{-\rho\langle uv\rangle}_{\text{molecular momentum transport}} = \frac{1}{2}\alpha\lambda\rho c \bar{U}_y$ $\quad$ c = molecular velocity

$\mu = \frac{1}{2}\alpha\lambda\rho c \propto \rho, c, \lambda$ $\quad$ λ = mean free path

Eddy viscosity: $\quad$ α = empirical constant < 1

$\sigma_t = -\frac{2}{3}\rho K I + \mu_t(\nabla\bar{U} + \nabla\bar{U}^T)$ in analogy

vs $\bar{\sigma} = -\bar{P} I + \mu(\nabla\bar{U} + \nabla\bar{U}^T)$

1D shear flow $\bar{U}(y)$: $\sigma_{t12}/\rho = -\overline{uv} = \nu_t \bar{U}_y$ $\quad$ $\nu_t = \mu_t/\rho$

(1) mixing length: $\mu_t = C_\mu \rho v L$ $\quad$ v plays role c

L plays role λ = mixing

$v = |L\bar{U}_y|$ $\quad$ local geometry length

$\mu_t = C_\mu \rho L^2 |\bar{U}_y|$ $\quad$ flow eg "distance wall"

(2) Local values K, ε: $L = K^{3/2}/\varepsilon$ $\quad$ $v = \sqrt{K}$

Eddy turn over time $\quad$ $\mu_t = C_\mu \rho K^2/\varepsilon$

$T_t = K/\varepsilon = l_e/u_{rms}$

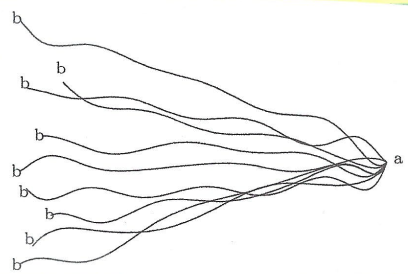
$\varepsilon = u^2_{rms}/l_e/u_{rms}$

$= u^3_{rms}/l_e$

$l_e = u^3_{rms}/\varepsilon$ $\quad$ $u_{rms} \sim \sqrt{K} = K^{1/2} \Rightarrow l_e = K^{3/2}/\varepsilon$

Lagrangian analysis turbulent transport

Figure 6.3 Ensemble of paths, each with a different initial position b, arriving at a.

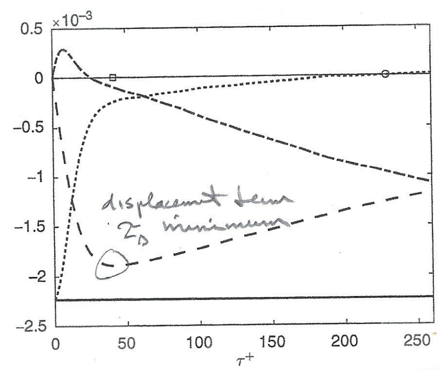


a - b = L

$$\overline{u_a v_a} = \underbrace{\overline{u_b v_a}}_{\substack{= \overline{u_a v_a} \text{ small } \tau \\ 0 \text{ large } \tau = \tau_D}} + \underbrace{\overline{v_a(\overline{U}_b - \overline{U}_a)}}_{\text{displacement}} + \underbrace{\overline{v_a(\overline{U}_a - \overline{U}_b)}}_{\text{transport due a → b}}$$

transport term: eddy carrying momentum over mixing time scale with surrounding fluid. If $\overline{U}$ linear = gradient model

Figure 6.4 Decomposition in Eq. (6.24) at $y^+ = 54.8$. $\cdots$, $\overline{u_b v_a}$ with zero point, τ_m, denoted by a circle; – –, $\overline{v_a(\overline{U}_b - \overline{U}_a)}$ with minimum denoted by a square; – · –, $\overline{v_a(U_a - U_b)}$; —, $\overline{u_a v_a}$.



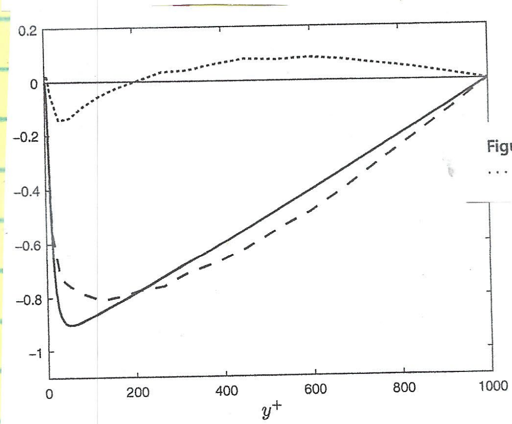


Figure 6.5 Evaluation of Eq. (6.24) at τ_D computed across the channel. —, $\overline{u_a v_a}$; – –, $\overline{(\overline{U}_b - \overline{U}_a)v_a}$; $\cdots$, $\overline{(U_a - U_b)v_a} + \overline{u_b v_a}$.

Displacement term most important $\overline{u_a v_a}$

Transport producing motions

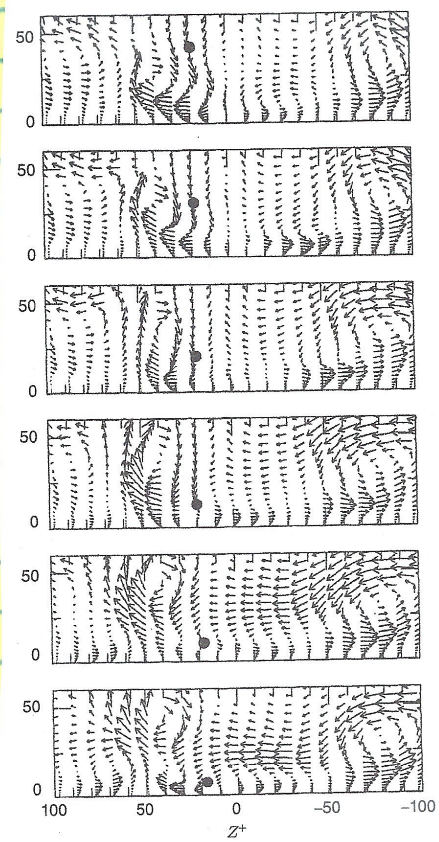


Figure 6.8 Fluid particle arriving at $y^+ = 7.3$ [7] due to a sweep event. Time increases moving from top to bottom image. Reprinted with the permission of Cambridge University Press.

and ejection events

$\overline{U}_b - \overline{U}_a = -L_2 \overline{U}_y$

$\langle v(\overline{U}_b - \overline{U}_a) \rangle =$

$-\overline{v L_2 \overline{U}_y} =$

$-\overline{v L_2}\, \overline{U}_y$

Gradient transport $\overline{U}_b = \overline{U}(a - L)$

$\approx \overline{U}_a - L_2 \overline{U}_y$

Substitute displacement transport term $\langle v(\overline{U}_b - \overline{U}_a) \rangle = -\overline{v L_2}\, \overline{U}_y$

v = velocity

$\tau_{T12}/\rho = -\overline{uv} = \nu_t \overline{U}_y \Rightarrow \nu_t = \overline{v L_2}$

L_2 = length scales

define Lagrangian auto correlation

$$\rho_{vv}(s) = \langle v(X(t),t)\, v(X(t+s),t+s)\rangle / \langle v(X(t),t)^2\rangle$$

$$\nu_t = \overline{v^2}\, T_{22} \qquad T_{22} = \int_{-\infty}^{0} \rho_{vv}(s)\, ds = \text{Lagrangian length scale}$$

vs.

$$L = \int_{t-\tau}^{t} v(X(s),s)\, ds \Rightarrow L_2$$

$$\nu_t = -\overline{uv}/U_y$$

$$\overline{v L_2(X,t)} = \int_{t-\tau}^{t} \overline{v(X,t)\, v(X(s),s)}\, ds$$

$$= \overline{v^2} \int_{t-\tau \to -\infty}^{t \to 0} \rho_{vv}(s)\, ds$$

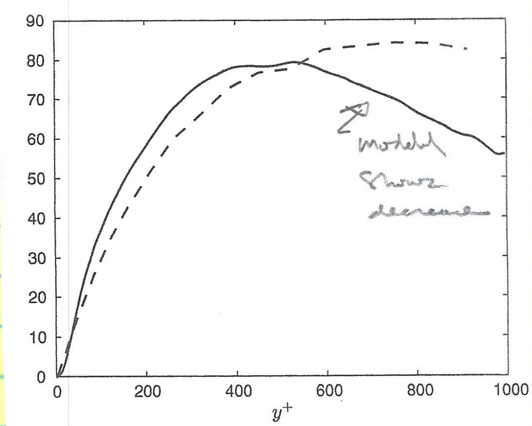


Figure 6.10 Eddy viscosity in channel flow: ––, $\mathcal{T}_{22}^+ \overline{v^2}^+$; —, ν_t^+.

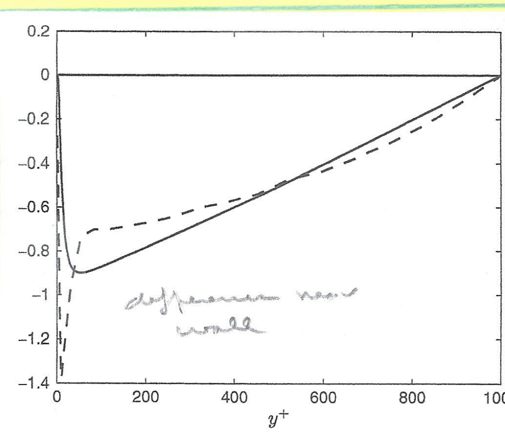


Figure 6.11 Inadequacy of gradient transport physics: —, $\overline{uv}^+$; ––, $-\mathcal{T}_{22}^+ \overline{v^2}^+ \, d\overline{U}^+/dy^+$.

Rough agreement with discrepancies due neglected effects

Homogeneous shear flow

$S = \bar{U}_y > 0$ superimposed on homogeneous/isotropic turbulence can be used to assess gradient transport model $\overline{u_a v_a} = \cancel{\overline{u_b v_a}} + \overline{v_a(\bar{U}_b - \bar{U}_a)}$

$+ \overline{v_a(\cancel{\bar{U}_b} - \bar{U}_a)}$

$\nu_t = \overline{v^2}\, T_{22}$ $\qquad = -\overline{v L_2}\, \bar{U}_y$

$T_{22} = \int_{-\infty}^{0} f_{vv}(s)\, ds$ $\qquad = \nu_t \bar{U}_y$

$f_{vv} = \langle v(\underline{X}(t), t)\, v(\underline{X}(t+s), t+s) \rangle / \langle v(\underline{X}(s), t)^2 \rangle$

(1) $K_t = P - \varepsilon \qquad P = -\overline{uv}\, S > 0$ since $\overline{uv} < 0$

$\varepsilon_t = P_\varepsilon^1 + P_\varepsilon^2 + P_\varepsilon^4 - Y_\varepsilon \qquad P_\varepsilon^1 = -\varepsilon_{ij}^c \bar{U}_{i,j}$

$P_\varepsilon^2 = -\varepsilon_{ij} \bar{U}_{i,j}$

$\omega_1 = w_y - v_z \qquad P_\varepsilon^1 + P_\varepsilon^2 = -2\nu \overline{\omega_1 \omega_2}\, S$

$\omega_2 = u_z - w_x \qquad$ using $\langle u_{i,m} u_{j,n} \rangle =$

$\langle u_{i,n} u_{j,m} \rangle$

Simple shear flows: $\dfrac{\overline{\omega_1 \omega_2}}{S} = \dfrac{\overline{uv}}{2K} \qquad \varepsilon = \nu S$

$P_\varepsilon^1 + P_\varepsilon^2 = C_{\varepsilon 1} P \varepsilon / K$

$P_\varepsilon^4 + Y_\varepsilon = S_K^* R_T^{1/2} \varepsilon^2/K - G^* \varepsilon^2/K \qquad G^* = (S_K^* - C_{\varepsilon 3})\sqrt{R_T} + C_{\varepsilon 2}$

Section 5.4

(2) $\varepsilon_t = C_{\varepsilon 1} P \varepsilon/K + C_{\varepsilon 3} R_T^{1/2} \varepsilon^2/K - C_{\varepsilon 2} \varepsilon^2/K \qquad C_{\varepsilon 3} = 0$ standard RANS model

(1) & (2) solved using model for P

Homogeneous shear flow is idealization but can be approximated by grid turbulence and DNS: K and ε exponential growth same rate for $St = \overline{U}_y t < 30 \Rightarrow SK/\varepsilon \approx 6$ and $P/\varepsilon \approx 1.8$

Self-similar ie when statistics normalized S and K are $\neq f(t)$!

Let $t^* = St \Rightarrow K^*(St) = K(t)/K(0)$

$$K^*_{t^*} = \varepsilon/SK\,(P/\varepsilon - 1)\,K^*$$

$$K^*(t^*) = e^{.13 t^*}$$

Long term behavior K and ε:

① $P = \varepsilon$ equilibrium

② exp growth continues

Must solve ① and ②: $\overline{v^2} = 2K/3$

$T_{22} \propto K/\varepsilon$ = eddy turnover time

$T_{22} = \frac{3}{2} c_\mu K/\varepsilon$

$$K_t = c_\mu K^2/\varepsilon\, S^2 - \varepsilon$$

$$\varepsilon_t = C_{\varepsilon 1} c_\mu K S^2 + C_{\varepsilon 3} R_T^{1/2} \frac{\varepsilon^2}{K} - C_{\varepsilon 2} \frac{\varepsilon^2}{K}$$

$$\nu_t = 2K/3 \times \tfrac{3}{2} c_\mu K/\varepsilon = c_\mu K^2/\varepsilon$$

$$\overline{uv} = -c_\mu K^2/\varepsilon\, S$$

A $C_{\varepsilon 3} = 0$ eliminates vortex stretching

B $C_{\varepsilon 3} = .1$ Bernard solutions

$C_{\varepsilon 1} = 1.45$

$c_\mu = .09$

$C_{\varepsilon 2} = 1.9$

Solution A = exponential growth

B = $P = \varepsilon$ = most likely physical outcome

Shows importance vortex stretching term

Grid turbulence : $\overline{v}_y = 0 \Rightarrow P = 0$ = model decaying homogeneous turbulence

v_0 = constant, M = mesh spacing

In frame of reference moving v_0, flow statistically stationary with statistics only $f(x)$ & turbulence (moving v_0) homogeneous & evolves with $t = x/v_0$

$\overline{u^2} > \overline{v^2} = \overline{w^2}$ $\sqrt{\overline{u^2}}$ is 10% greater $\sqrt{\overline{v^2}}$ but can be made equal

$$k/v_0^2 \propto (x/M)^{-1.3}$$

or $$k/v_0^2 = A\left(\frac{x-x_0}{M}\right)^{-n} \qquad 1.15 \le n \le 1.45$$

in moving frame $k(t) = k_0 \left(t/t_0\right)^{-n}$

$$k_t = -\left(\frac{n k_0}{t_0}\right)\left(t/t_0\right)^{-(n+1)}$$

$$= -\varepsilon$$

$$\varepsilon = \varepsilon_0 \left(t/t_0\right)^{-(n+1)}$$

As turbulence decays $Re \downarrow \Rightarrow n = 5/2$

Final period decay

Vorticity Transport

$$\bar U_{i,t} + \bar U_j \bar U_{i,j} = -\nabla(P/\rho + k) + \nu \nabla^2 \bar U_i + \overline{\underline{u} \times \underline{\omega}}$$

$$\langle \underline{u} \times \underline{\omega} \rangle_i = \varepsilon_{ijk} \overline{u_i \omega_j} = \text{vortex flux correlation}$$

Alternative formulation, with idea $\overline{u_i \omega_j}$ more amendable model than $\overline{u_i u_j}$, as less sensitive P and $\nu \nabla^2 \bar U_i$.

Following same approach as used for momentum transport

$$\langle u_i^a \omega_j^a \rangle = \langle u_i^a (\bar\Omega_j^b - \bar\Omega_j^a) \rangle + \int_{t-\tau}^{t} \langle u_i \Omega_k(s) U_{j,k}(s) \rangle ds + \int_{t-\tau}^{t} \langle \nu u_i \nabla^2 \Omega_j(s) \rangle ds$$

τ large such that $\langle u_i^a \omega_j^b \rangle \approx 0$

$$\langle u_i \omega_j \rangle = \underbrace{- \int_{t-\tau}^{t} \langle u_i u_k(s) \rangle ds \, \bar\Omega_{j,k}}_{\text{displacement transport using TS for } \bar\Omega_j^b} + \underbrace{\int_{t-\tau}^{t} \langle u_i u_{j,k}(s) \rangle ds \, \bar\Omega_k}_{\text{stretching term after using Reynolds decomposition}}$$

neglects viscous term

Evaluated using channel flow DNS: $\bar U(y)$, $\bar\Omega_3 = -\bar U_y$ and Lagrangian integral scales. Shows accuracy $\langle u_i \omega_j \rangle$ equation physics

3.2 Closure problem and arrow of time

Closure Problem

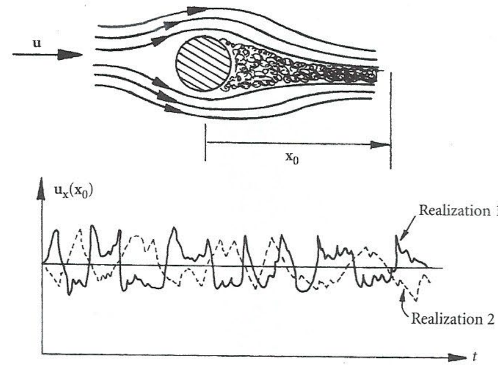


Figure 1.8 A cylinder is towed through a quiescent fluid and $\mathbf{u}_x$ is measured at location x_0. The records of $\mathbf{u}_x(t)$ in two nominally identical realizations of the experiment are quite different.

$\underline{u}(\underline{x}, t)$ random

$\bar{u}(\underline{x})$ and $\overline{u^2}(x)$ reproducible ie statistical properties should be predictable

∴ How can equations be derived for statistical properties

$$\rho \underline{u}_t = -\rho \underline{u} \cdot \nabla \underline{u} - \nabla p + \mu \nabla^2 \underline{u} \qquad (1)$$

$$\underline{u}_t = F_1(\underline{u}, p)$$

$$\nabla \cdot \underline{u} = 0$$

$$\nabla \cdot (1) \Rightarrow \quad \nabla^2 (p/\rho) = -\nabla \cdot (\underline{u} \nabla \underline{u})$$

$$p(\underline{x}) = \frac{\rho}{2\pi} \int \frac{[\nabla \cdot (\underline{u} \cdot \nabla \underline{u})]'}{|x - x'|} dx'$$

Biot-Savart Law

ie $\underline{u}_t = F_2(\underline{u})$ for IC integrate in time "deterministically" find $u(\underline{x}, t)$

DNS!

However, impracticable for industrial use and often statistical variables of greatest interest such as $\bar{u}$, $\overline{u'^2}$ etc. [$\underline{u}(\underline{x},t) = \underline{u}(x) + \underline{u}'(\underline{x},t)$]

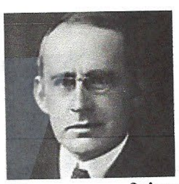
Need equations $\bar{u}$, $\overline{u'^2}$ etc. However turns out impossible to derive equations for statistical variables as leads to hierarchy of statistical non deterministic equations each depending on higher order statistical variables

$\frac{\partial}{\partial t}$ (statistical property $\underline{u}$) = F(order higher order statistical properties $\underline{u}$)

"Closure problem"

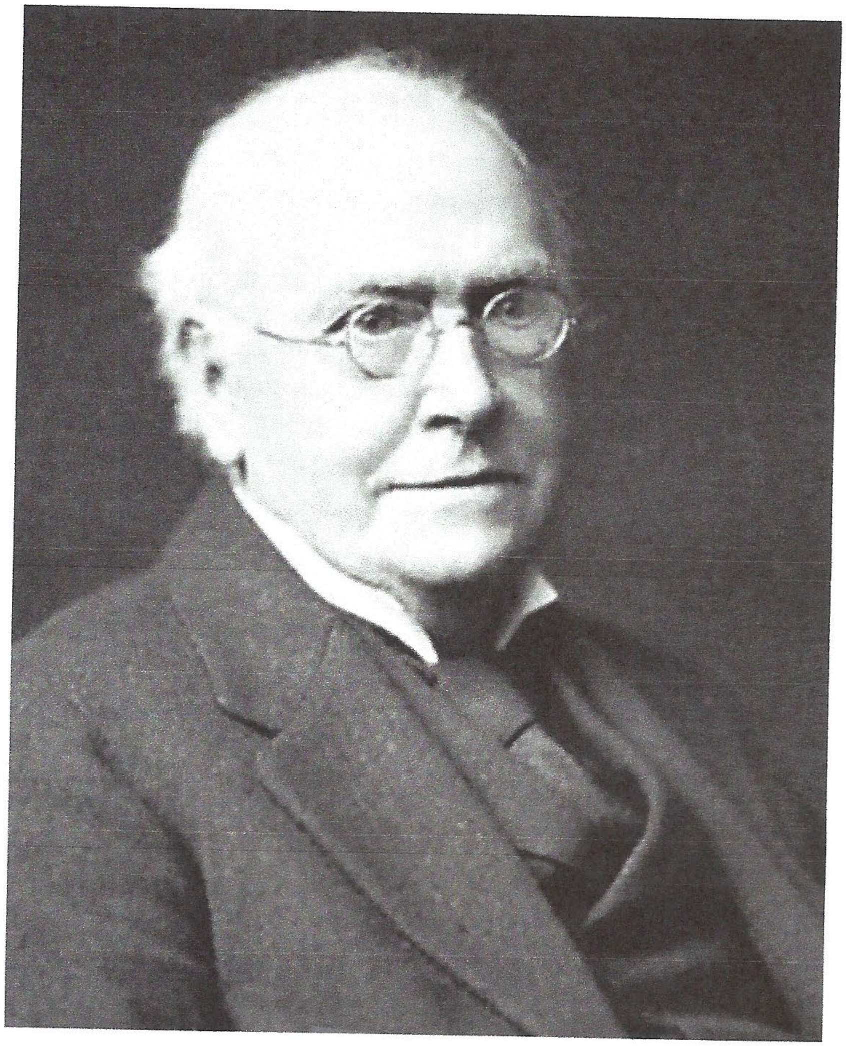
Arrow of Time

Arrow of Time



The arrow of time, also called time's arrow, is the concept positing the "one-way direction" or "asymmetry" of time. It was developed in 1927 by the British astrophysicist Arthur Eddington and is an unsolved general physics question. This direction, according to Eddington, could be determined by studying the organization of atoms, molecules, and bodies, and might be drawn upon a four-dimensional relativistic map of the world ("a solid block of paper").

Euler equation $\underline{u}_t + \underline{u}\cdot\nabla\underline{u} = -\nabla(p/\rho)$ is time reversable, ie, replace t and $\underline{u}$ by $-t$ and $-u$ and can integrate backwards to obtain IC. However due to effects of entropy NS equations are not time reversable!



SIR HORACE LAMB

Sir Horace Lamb (1849–1934) is best known for his extremely thorough and well-written book, *Hydrodynamics*, which first appeared in 1879 and has been reprinted numerous times. It still serves as a compendium of useful information as well as the source for a great number of papers and books. If this present book has but a small fraction of the appeal of *Hydrodynamics*, the authors would be well satisfied.

Sir Horace Lamb was born in Stockport, England in 1849, educated at Owens College, Manchester, and then Trinity College, Cambridge University, where he studied with professors such as J. Clerk Maxwell and G. G. Stokes. After his graduation, he lectured at Trinity (1822–1825) and then moved to Adelaide, Australia, to become Professor of Mathematics.

After ten years, he returned to Owens College (part of Victoria University of Manchester) as Professor of Pure Mathematics; he remained until 1920.

Professor Lamb was noted for his excellent teaching and writing abilities. In response to a student tribute on the occasion of his eightieth birthday, he replied: "I did try to make things clear, first to myself . . . and then to my students, and somehow make these dry bones live."

His research areas encompassed tides, waves, and earthquake properties as well as mathematics.