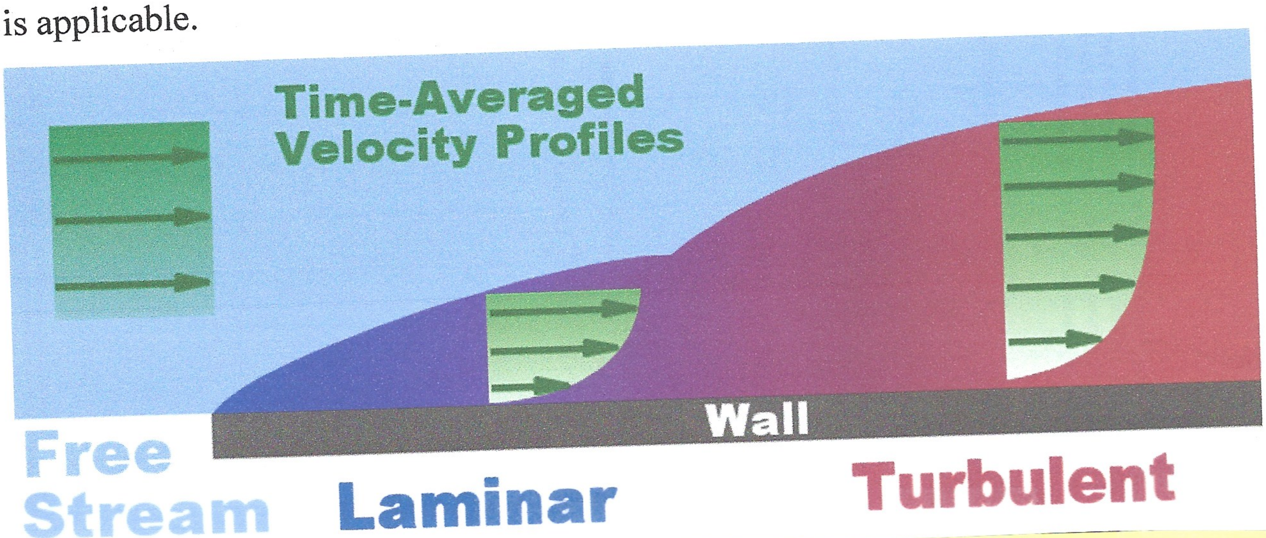
Chapter 8 Boundary Layers

Consider high Re flow (1) around streamlined/slender body for which viscous effects are confined to a narrow boundary layer near the solid surface/wall or (2) for free shear flows, i.e., jets, wakes and mixing layers for which the vorticity is similarly confined to a narrow region. In both cases Prandtl's boundary layer theory is applicable.



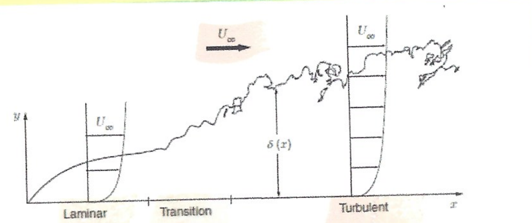


Figure 8.1 Turbulent boundary layer over a flat plate.

Figure 8.2 Smoke visualization of a turbulent boundary layer at $R_\theta = 3000$ [1].

Inner (viscous) and outer (inviscid; often potential) flow divided sharp corrugated interface defined by intermittency function

Stability and transition: arguably top of list of difficult/complex fluid mechanics problems still at fore front of research

$Re_{x\,crit} \sim 4 \times 10^5 \quad Re_x = U_\infty x / \nu$

1. Stable laminar flow near the leading edge.
2. Unstable two-dimensional Tollmien–Schlichting waves.
3. Development of three-dimensional unstable waves and hairpin eddies.
4. Vortex breakdown at regions of high localized shear.
5. Cascading vortex breakdown into fully three-dimensional fluctuations.
6. Formation of turbulent spots at locally intense fluctuations.
7. Coalescence of spots into fully turbulent flow.

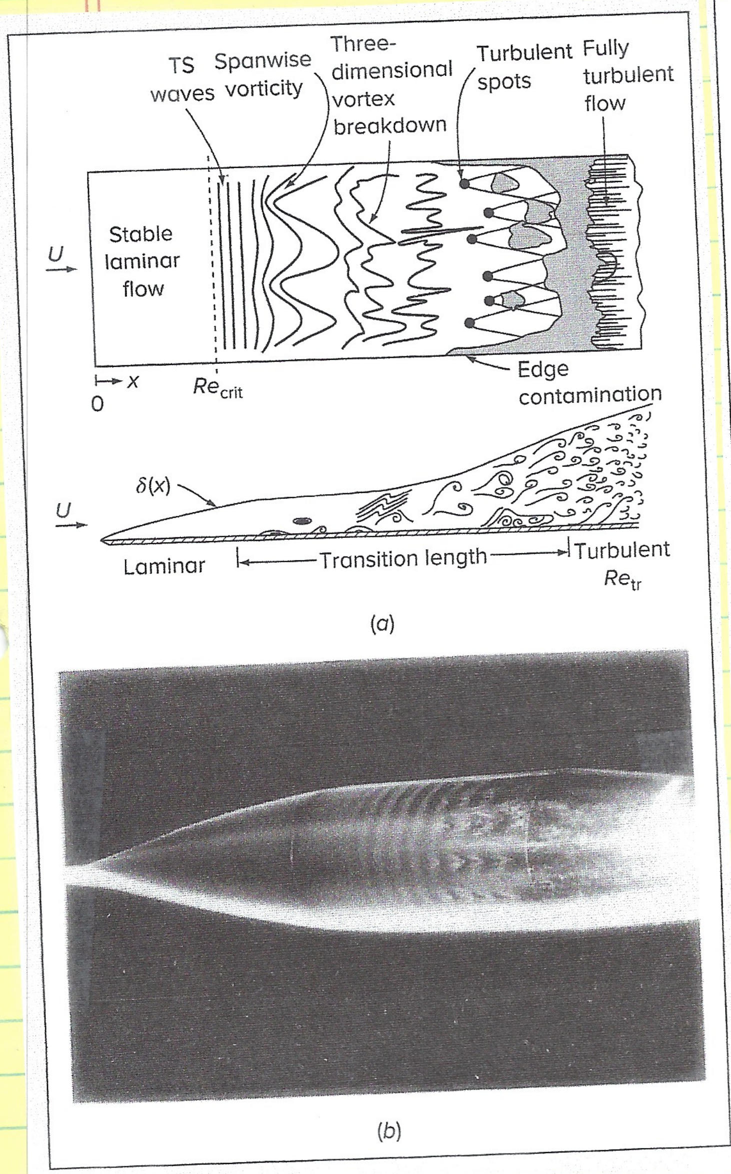


FIGURE 5-28
Description of the boundary-layer transition process: (*a*) idealized sketch of flat-plate flow and (*b*) smoke visualization of flow with transition induced early by acoustic input at $Re_L =$ 814,000 and 500 Hz. [*Courtesy of J.T. Kegelman and T.J Mueller, University of Notre Dame*].

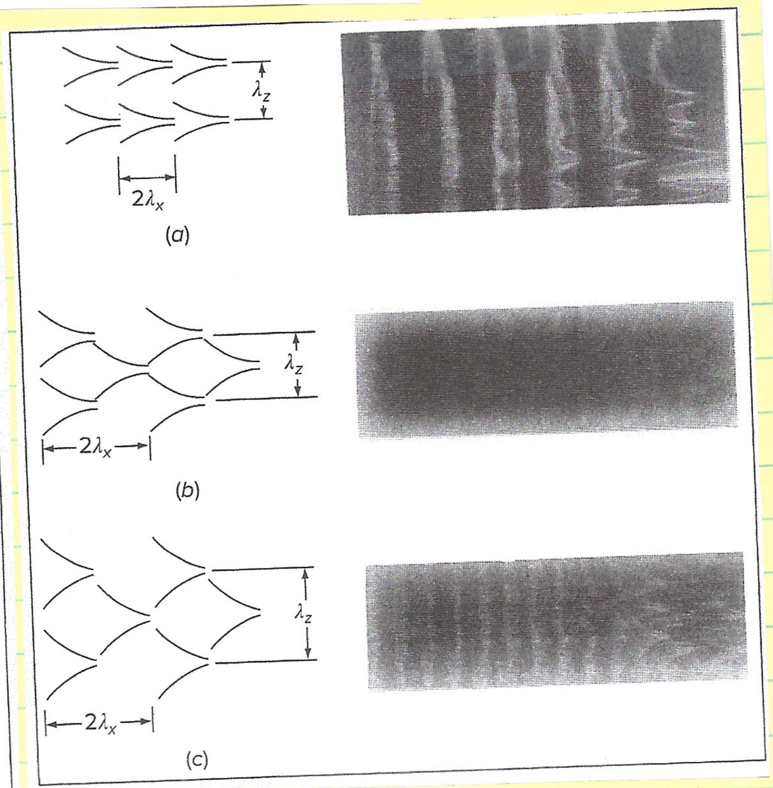


FIGURE 5-26
Patterns of unstable vortex breakdown in a boundary layer: (*a*) K-type ($u'/U_\infty \approx 1$ percent) aligned in phase and similar to a Tollmien–Schlichting wave; (*b*) C-type (0.3 percent) staggered subharmonic with $\lambda_z \approx 1.5\lambda_x$; (*c*) H-type (0.6 percent) staggered subharmonic with $\lambda_z \approx 0.7\lambda_x$. [*Courtesy of Dr. William S. Saric*].

8.1 General Properties

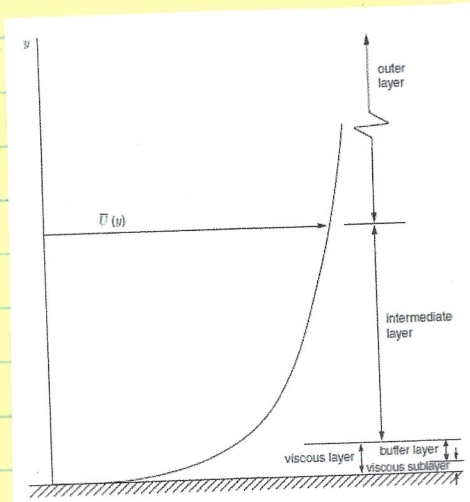


Figure 8.3 Boundary layer zones, not drawn to scale.

Common structure other wall flows (eg, channel & pipe), except stronger influence of departure from log law outer layer. Also $U(x)$ not fixed as per h or D.

arbitrary

$$\overline{U}(x,\delta) = \boxed{.99}\, U_\infty(x)$$

δ^*: displacement thickness (δ_1)

Outer Flow

$$\int_0^\infty \overline{U}(x,y)\,dy = \int_{\delta_1}^\infty U_\infty(x)\,dy \qquad U_\infty^2/2 + \overline{P}/\rho = U_0^2/2 + P_0/\rho$$

Equivalent discharge

= reference values

$$\delta_1(x) = \int_0^\infty \left(1 - \overline{U}(x,y)/U_\infty(x)\right) dy \qquad \overline{P}_x = -\rho U_\infty U_{\infty x}$$

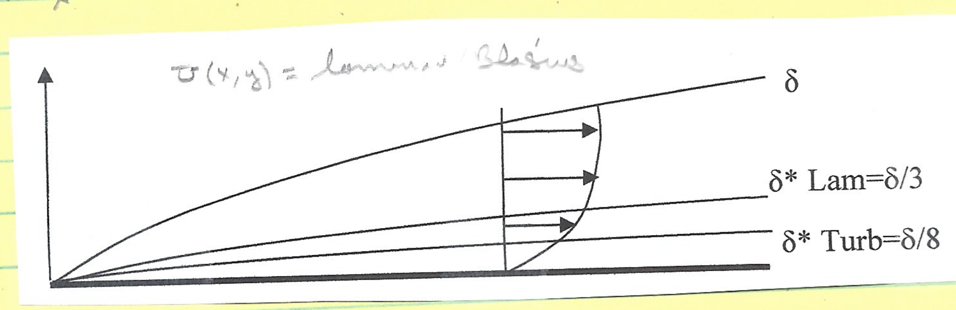
Measure of distance outer flow displaced by BL

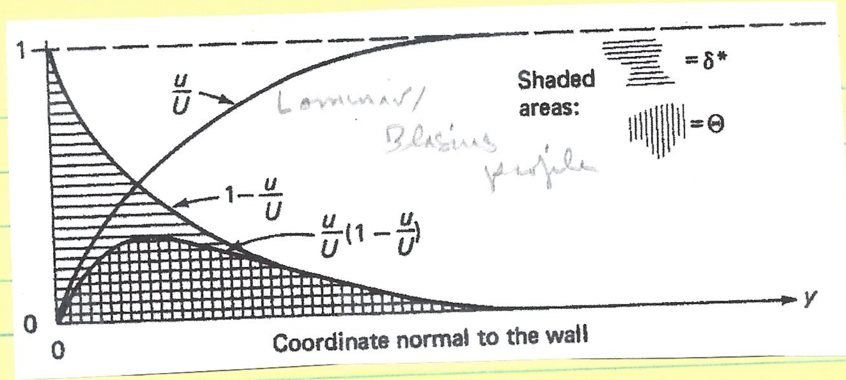
θ: momentum thickness

$$\theta(x) = \int_0^\infty \frac{\overline{U}(x,y)}{U_\infty(x)}\left(1 - \frac{\overline{U}(x,y)}{U_\infty(x)}\right) dy$$

measure loss of momentum due BL

$Re_x = U_\infty x/\nu \quad R_\delta = U_\infty \delta/\nu \quad R_\theta = U_\infty \theta/\nu$





Intermittency

$\delta(x)$ = mean BL thickness

In reality intermittent

γ = fraction of time flow at y/δ is turbulent

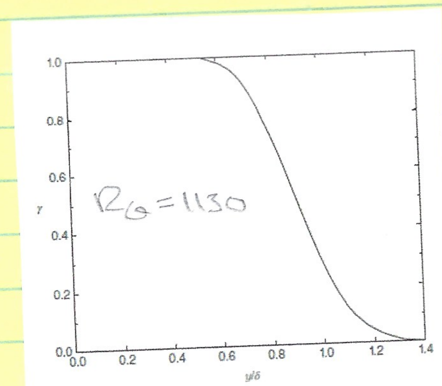


Figure 8.4 Intermittency factor in a turbulent boundary layer [5].

$\gamma(y) = \frac{1}{2}(1-\mathrm{erf}(y/\delta - .78))$ $\bar{P}_x = 0$

$\bar{y} = .78\delta \pm .14\delta$

$\gamma(y) = 1/(1+5.5(y/\Delta)^6)$

$\Delta = f(\text{flow})$

8.2 BL Growth

Blasius: $\theta/x = .664/Re_x \Rightarrow \theta \propto \sqrt{x}$

$F_x = .664\,\rho U_\infty^2 L/Re_L$ per unit width

Turbulent Flow: $\bar{U}^2_x + (\bar{U}\bar{V})_y = -\rho^{-1}\bar{P}_x + \nu\bar{U}_{yy} - \overline{uv}_y$

$(\bar{P}/\rho + \overline{v^2})_y = 0$

$\bar{P}_x = -\rho U_\infty U_{\infty x}$

$\frac{\partial}{\partial x}\left(\bar P/\rho + \overline{v'^2}\right)_y = \frac{\partial}{\partial y}\bar P_x \qquad \overline{v'^2}_x = 0$

ie $\bar P_x = -\rho U_\infty U_{\infty x}$ applies across BL

$y = d =$ free stream where $\bar U = U_\infty$

$$\frac{\partial}{\partial x}\int_0^d \bar U^2\,dy + U_\infty \bar V(x,d) = \int_0^d U_\infty U_{\infty x}\,dy - \tau_w/\rho \qquad \left.\overline{u'v'}\right|_0^d = 0$$

Continuity $\bar V(x,d) = -\int_0^d \bar U_x\,dy$

Combining: $\frac{d}{dx}\int_0^d \left(\bar U^2 - \bar U U_\infty\right)dy = U_{\infty x}\int_0^d \left(U_\infty - \bar U\right)dy - \tau_w/\rho$

letting $d \to \infty$ where terms in ( ) = 0

$$\frac{d}{dx}\left(U_\infty^2\theta\right) + U_{\infty x}U_\infty\delta_1 = \tau_w/\rho$$ momentum integral equation BL with $\bar P_x$

ie $\tau_w(x) = f(\delta_1, \theta)$

Assume power law velocity profile: $\bar U = U_\infty(x)\left(\frac{y}{\delta}\right)^{1/n}$

$\Rightarrow \delta_1 = \delta/1+n \qquad \theta = \delta n/(n+1)(n+2)$

For $n = 7$ $\quad \frac{7}{72}\frac{d}{dx}\left(U_\infty^2\delta\right) + \frac{1}{8}\frac{dU_\infty}{dx}U_\infty\delta = \tau_w/\rho$

need $\tau_w = f(\delta)$ Power law fit data

For U_∞ = constant: $\tau_w/\rho = .0225\frac{\nu^2}{\delta^2}Re_\delta^{7/4}$

$\Rightarrow \frac{d\delta}{dx} = .0225\frac{72}{7}Re_\delta^{-1/4}$

ie $\delta/x = .37\,Re_x^{-1/5} \qquad \delta \propto x^{4/5} \gg x^{1/2}$ laminar flow

OK for $Re < 10^6$; for higher Re use $n = 8$ or higher

8.3 Log-law behavior of mean velocity and variance

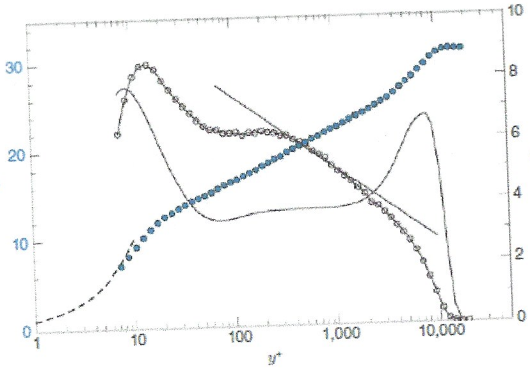


Figure 8.5 Mean and variance of the streamwise velocity in boundary layer flow at $R_\tau = 13{,}600$. •, $\overline{U}$; —, $y^+\partial\overline{U}^+/\partial y$; o, $\overline{u^2}^+$; straight line is a fit to Eq. (8.25). Data from [11]. Figure reproduced from [12] by permission of Annual Reviews.

$R_\tau = u_\tau \delta / \nu = 13{,}600$

$u^+ = \kappa^{-1} \ln y^+ + B$

$\kappa = .41 \quad B = 5$

log law region

$30 \le y^+ \le .15\delta$

$B = y^+ u^+_{y^+}$

$1{,}000 \le y^+$ rises slightly until about 2500

Similar behavior channel and pipe flow with primary difference due outer layer

Combining all three flows, log law found for $3\sqrt{R_\tau} \le y^+ \le .15 R_\tau$ $\quad \kappa = .39 \quad B = 4.3$

Attached eddy hypothesis: Townsend

$\overline{u^2}^+ = B_1 - A_1 \ln(y/\delta)$ $\qquad A_1 = 2.39$

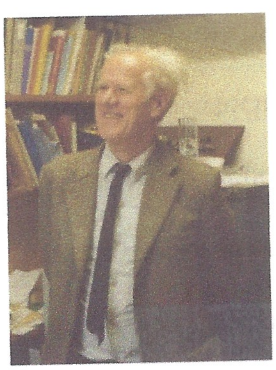
$B_1 = 1.03$

in outer part log law region

Townsend (1976), Page 153:

It is difficult to imagine how the presence of the wall could impose a dissipation length-scale proportional to distance from it unless the main eddies of the flow have diameters proportional to distance of their 'centres' from the wall because their motion is directly influenced by its presence. In other words, the motion is directly influenced by its presence. In other words, the velocity fields of the main eddies, regarded as persistent, organised flow patterns, extend to the wall and, in a sense, they are *attached* to the wall. We proceed to consider the observed characteristics of a motion made up from the superposition of attached eddies of a wide range of sizes.

Let us suppose that the main, energy-containing motion is made up of contributions from 'attached' eddies with similar velocity distributions,



8.4 Outer layer

Intermittency distinguishes BL outer region from channel & pipe flows, which reach fully developed condition vs BL spatially develop in streamwise direction. Also $\bar{V} \neq 0$ since $\bar{U}_x \neq 0$.

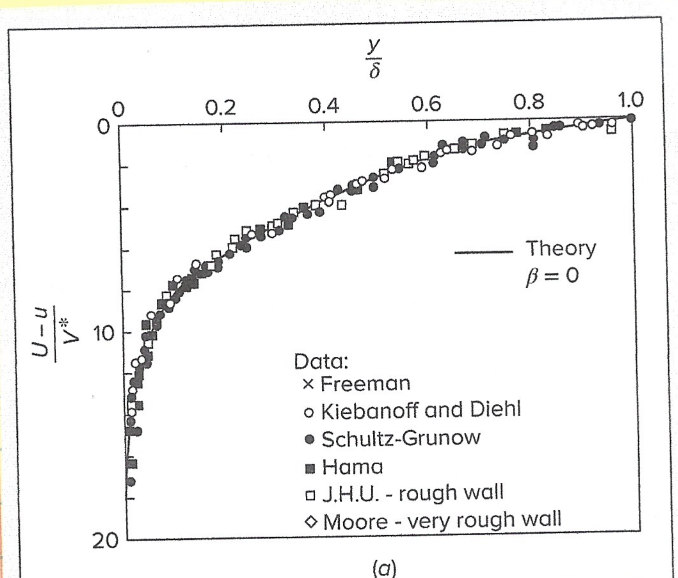
Outer flow scaling: $U_\infty - \bar{U}(y) = f(y, \delta, \rho, u_\tau, P_{\infty x})$
$\neq f(\nu)$

Dimensional analysis:

$$(U_\infty - \bar{U}(y))/u_\tau = U_\infty^+ - \bar{U}^+ = F\Big(y/\delta, \underbrace{\tfrac{\delta}{\rho u_\tau^2} P_{\infty x}}\Big)$$

$U_\infty^+ - \bar{U}^+$ for BL different $P_{\infty x}$ but same β collapse same profile

if δ replaced δ_1 = Clauser equilibrium parameter β



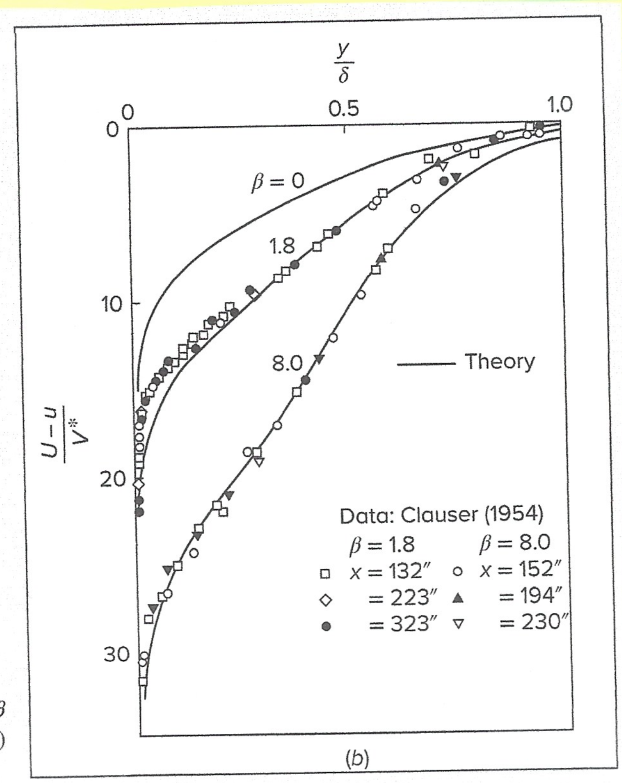


FIGURE 6-12
Equilibrium-defect profiles, as correlated by the Clauser parameter β and the theory of Mellor and Gibson (1966): (a) flat-plate data; (b) equilibrium adverse gradients.

$\bar{U}(y)$ deviation log law for $.15 \le y/\delta \le 1$

$$\kappa^{-1}\ln y^+ + B = U_\infty^+ - F(y/\delta)$$

$F(y/\delta)$ = difference between outer flow U_∞^+ & log law

$y \ll \delta$
ie overlap region
as per 7.2.1

$$F(y/\delta) = U_\infty^+ - \kappa^{-1}\ln y^+ + B = [U_\infty^+ - \kappa^{-1}\ln\delta^+ - B] - \kappa^{-1}\ln(y/\delta)$$

For y/δ in both overlap & outer regions include wake function $W(y/\delta)$

$$F(y/\delta) = [U_\infty^+ - \kappa^{-1}\ln\delta^+ - B] - \kappa^{-1}\ln(y/\delta) - \frac{\Pi}{\kappa} W(y/\delta)$$

$\Pi\kappa^{-1}W(y/\delta)$ = amount $\bar{U}^+$ above log law in outer region $y > .15\delta$ & $= 0$ in log law region. Since $F(1) = 0$

$$W(1)\,\Pi\kappa^{-1} = U_\infty^+ - \kappa^{-1}\ln\delta^+ - B$$

$$\Rightarrow \bar{U}^+ = \kappa^{-1}\ln y^+ + B + \frac{\Pi}{\kappa} W(y/\delta)$$

$$= \kappa^{-1}\ln y^+ + B + \frac{W(y/\delta)}{W(1)}\left(U_\infty^+ - \kappa^{-1}\ln\delta^+ - B\right)$$

ie $\dfrac{W(y/\delta)}{W(1)} = \dfrac{\bar{U}^+ - \kappa^{-1}\ln y^+ - B}{U_\infty^+ - \kappa^{-1}\ln\delta^+ - B}$ = fractional velocity defect relative log law

Emerical models:

$$\Pi \kappa^{-1} W(y/\delta) = \frac{2\Pi}{\kappa} \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

or $$= \kappa^{-1}(1+6\Pi)(y/\delta)^2 - \kappa^{-1}(1+4\Pi)(y/\delta)^3$$

where $W(1) = 2$

$$\Pi = f(U_\infty^+, \delta^+) = f(x)$$

For $P_{\infty x} = 0$

$$Re_\delta = \delta U_\infty/\nu = .37\, Re_x^{4/5} \qquad \delta/x = .37\, Re_x^{-1/5} \qquad Re_x = U_\infty x/\nu$$

$$u_\tau^2 = \tau_w/\rho = .0225 \frac{\nu^2}{\delta^2} Re_\delta^{7/4} \qquad \delta = .37\, x\, Re_x^{-1/5}$$

$$\frac{U_\infty \delta}{\nu} = .37 \frac{U_\infty x}{\nu} Re_x^{-1/5}$$

$$U_\infty^+ = U_\infty/u_\tau = 5.89\, Re_x^{1/10}$$

$$\delta^+ = .0628\, Re_x^{7/10}$$

$$W(1)\Pi\kappa^{-1} = U_\infty^+ - \kappa^{-1}\ln\delta^+ - B$$

$$\Pi = \frac{\kappa}{2}\left(5.89(Re_x)^{1/10} - \kappa^{-1}(-2.77 + .7\ln(Re_x)) - B\right)$$

which is slowly varying in x, e.g., with $\kappa = .4$, $B = 5.1$ Re_x ranges from 5×10^6 to 10^7 and Π varies from .48 to .63. Typically $\Pi = .55$ is used